NASA TT F-30

N-34
39683

TECHNICAL TRANSLATION

F-30

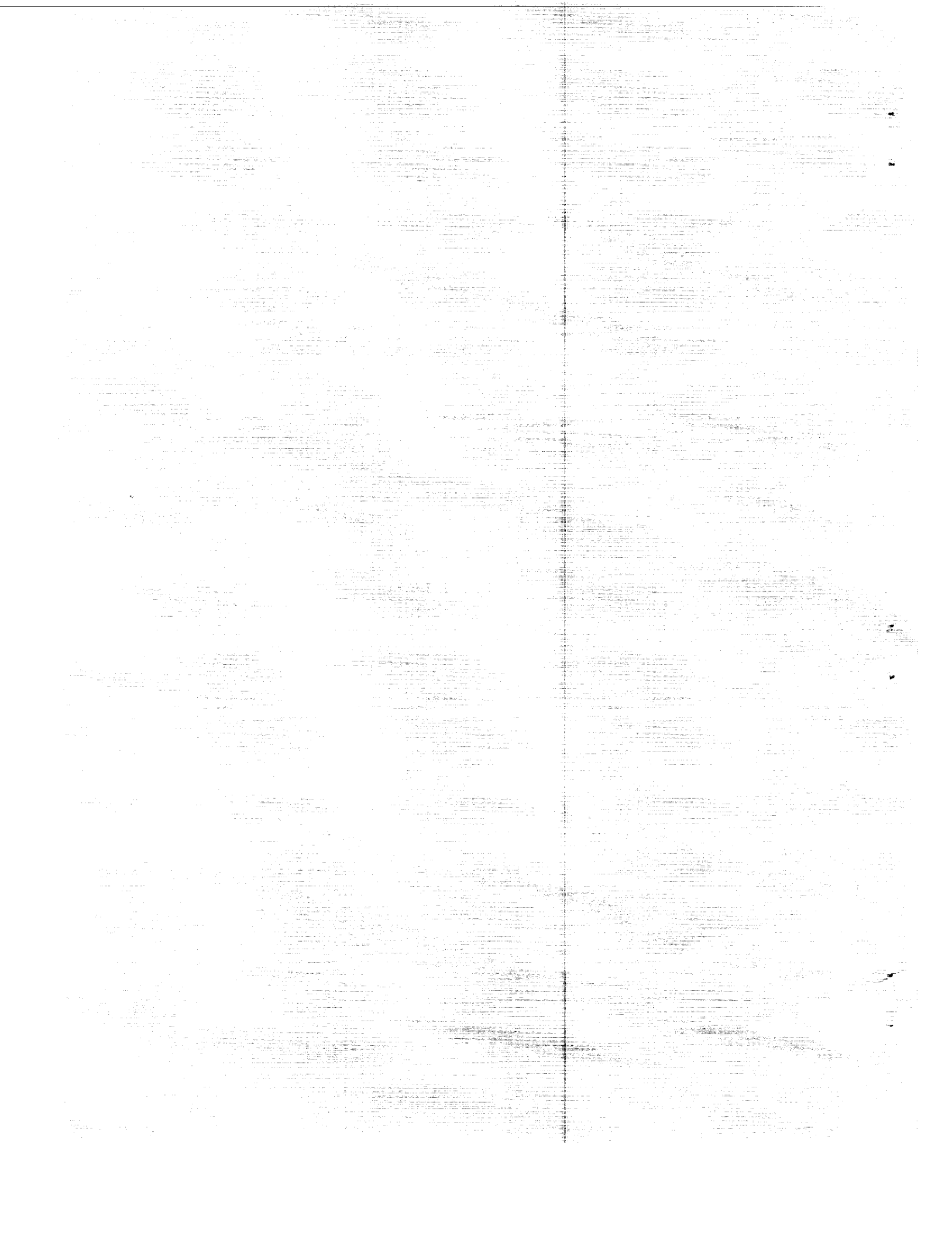
"SIMILAR" THREE-DIMENSIONAL BOUNDARY LAYERS

By Theo Geis

Translation of " 'Ähnliche' dreidimensionale Grenzschichten." Inaugural dissertation for obtaining the degree of Doctor of the Faculty of Natural Sciences and Mathematics of the Albert-Ludwigs University at Freiburg im Breisgau, 1955.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON March 1961



NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL TRANSLATION F-30

"SIMILAR" THREE-DIMENSIONAL BOUNDARY LAYERS*

By Theo Geis

1. INTRODUCTORY ABSTRACT

L
4
0
0

In two-dimensional boundary-layer flows, almost exclusively considered up to the present time, all flow parameters are independent of the third coordinate. In three-dimensional flow processes, in contrast, the outer potential flow is a function of two space coordinates parallel to the wall surrounded by the flow, whereas the boundary-layer flow shows all three velocity components which, in the most general case, are dependent on all three space coordinates.

Besides the addition of a new coordinate and velocity component, several novel phenomena arise in the three-dimensional case: If, for instance, the streamlines of the outer potential flow are rectilinear but convergent or divergent, there occurs, essentially, only a variation of the boundary-layer thickness, compared to the two-dimensional flow. In the case of lateral curvature of the outer potential streamlines, in contrast, a so-called "secondary flow" appears. The reason for the latter is: In the external flow the pressure gradient normal to the flow direction and the centrifugal force are in equilibrium. The same pressure gradient prevails inside the boundary layer as outside, whereas the centrifugal force is reduced due to the lower velocity so that the fluid particles can more easily yield to the transverse-pressure gradient. Thus a transport of fluid toward the concave side of the potential streamlines is produced. The appearance of this "secondary" or "transverse flow" causes the direction of the streamlines directly at the wall to deviate sometimes quite considerably from the direction of the outer potential streamlines.

The literature on three-dimensional boundary layers published up until now may be divided into two groups:

The first group includes reports dealing with the derivation of the three-dimensional boundary-layer equations and the discussion of these equations was well as with attempts at their simplification; foremost in

---

*"'Ähnliche' dreidimensionale Grenzschichten." Inaugural dissertation for obtaining the degree of Doctor of the Faculty of Natural Sciences and Mathematics of the Albert-Ludwigs University at Freiburg im Breisgau, 1955.

that group, are the reports of Hayes (ref. 1), Howarth (ref. 2), and Moore (ref. 3). The first-named author strives for a simplification of the boundary-layer equations by suitable selection of the coordinate system. Furthermore, he carries out a linearization of the boundary-layer equations with respect to the secondary flow. In Moore's report, which is restricted to the special case of rectangular Cartesian coordinates, a certain simplification is attained by introduction of a pair of stream functions in such a manner that the continuity equation is identically fulfilled.

The second group contains reports which consider special three-dimensional boundary-layer flows and actually solve the pertaining equations for these cases. Here, above all, must be mentioned a series of reports which treat the case of a cylinder in oblique flow. (See, for instance, ref. 4 which has a detailed list of references.) Here, limited to the incompressible case, the calculation of the boundary layer becomes particularly simple since the so-called "independence principle" is valid; this principle states that the system of boundary-layer equations consists of two equations for the primary flow component u and the component w perpendicular to the surface in the flow, which can be solved beforehand according to two-dimensional methods, and a third equation from which the secondary flow component v can be subsequently calculated. Likewise, the case of a flow about rotationally symmetrical bodies has been treated in detail. If the approach flow is axially symmetrical, the boundary-layer calculation may be reduced completely to the two-dimensional case, as Mangler (ref. 5) has shown. Other reports that should be named here are those of Bödewadt (ref. 6), Cochran (ref. 7), Hannah (ref. 8), Howarth (refs. 9 and 10), Moore (refs. 11, 12, 13, and 14), Loos (ref. 15), Mager (ref. 16), Mager and Hansen (ref. 17), and Wilkinson (ref. 18) which also deal with cases where some kind of symmetry occurs, or where a certain characteristic parameter which is small appears in the boundary-layer equations.

Among the flows treated in the last-named reports are several cases where the boundary-layer calculation leads to ordinary differential equations. This suggests the question as to whether it is possible to indicate - corresponding to the steady boundary-layer equations for the two-dimensional problem - a general class of three-dimensional boundary-layer flows whose calculation can be reduced to the solution of boundary-value problems of ordinary differential equations. The present report deals with this question.

This investigation was suggested by Professor Dr. H. Görtler. The author of this report wishes to express his deep gratitude to Professor Görtler for many fruitful discussions and for his vivid interest in the progress of the work. Likewise, the author is indebted to Mrs. W. Witting who, with much industry and patience, carried out the numerical calculations.

2. THE THREE-DIMENSIONAL BOUNDARY-LAYER EQUATIONS

It is expedient for the consideration of three-dimensional boundary-layer flows to use curvilinear space coordinates, which will be denoted below by x, y, and z, in such a manner that the surface F in the flow coincides with a coordinate surface $z = \text{Constant}$. Then, x and y are Gauss parameters on F so that the square of the arc element on the surface F, after introduction of the metric fundamental quantities $g^*_{ik}(x,y)$, reads

$$ds^{*2} = g^*_{11}\,dx^2 + 2g^*_{12}\,dx\,dy + g^*_{22}\,dy^2 \tag{2.1}$$

If coordinate systems having the parameter lines on F forming an orthogonal grid are used exclusively, the square of the arc element is reduced to the form:

$$ds^{*2} = g^*_{11}\,dx^2 + g^*_{22}\,dy^2 \tag{2.2}$$

In addition, it is assumed that the coordinate surfaces $x = \text{Constant}$ and $y = \text{Constant}$ perpendicularly intersect the surface F. This does not cause the coordinate system to be orthogonal also outside F. In an orthogonal space coordinate system, the three families of coordinate surfaces $x = \text{Constant}$, $y = \text{Constant}$, and $z = \text{Constant}$ intersect one another along the lines of curvature, according to the well-known theorem of Dupin. Since the intersections of the surfaces $x = \text{Constant}$ and $y = \text{Constant}$ with F are precisely the parameter lines on F, it is recognized that the grid of the parameter lines on F can be continued into an orthogonal system in space only when the parameter lines are, at the same time, lines of curvature.

In the space outside the surface in the flow, the square of the arc element has, therefore, the form

$$ds^2 = g_{11}\,dx^2 + 2g_{12}\,dx\,dy + g_{22}\,dy^2 + g_{33}\,dz^2 + 2g_{13}\,dx\,dz + 2g_{23}\,dy\,dz$$

On the surface F itself, which will be characterized by $z = 0$, ds^2 must be reduced to ds^{*2}; that is,

$$g_{11}(x,y,0) = g^*_{11}(x,y)$$

$$g_{22}(x,y,o) = g^*_{22}(x,y)$$

$$g_{12}(x,y,o) = 0$$

must be valid. Since the z-lines, moreover, had to stand perpendicularly on F, it must follow that

$$g_{13}(x,y,o) = g_{23}(x,y,c) = 0$$

Thus only the function $g_{33}(x,y,z)$ remains completely arbitrary.

In boundary-layer theory, it is assumed that the g_{ik} within the very thin boundary layer may be replaced by their values on the surface in the flow F. On the basis of this assumption, the square of the arc element then reads

$$ds^2 = g^*_{11}\, dx^2 + g^*_{22}\, dy^2 + g_{33}\, dz^2 \tag{2.3}$$

Within the boundary layer, the coordinate system, in particular, within the scope of the boundary-layer approximations may be regarded as orthogonal.

For what follows, it is expedient to use, instead of the metric fundamental quantities g_{ik} themselves, the functions

$$g_1 = \sqrt{g^*_{11}} \qquad g_2 = \sqrt{g^*_{22}} \qquad g_3 = \sqrt{g_{33}} \tag{2.4}$$

Thus,

$$ds^2 = g_1{}^2 dx^2 + g_2{}^2 dy^2 + g_3{}^2 dz^2 \tag{2.5}$$

where g_1 and g_2 are functions of x and y, whereas g_3 is an arbitrary function of x, y, z and may be chosen especially as a function of x and y alone if that is suitable. This arbitrariness in the selection of g_3 permits in special cases a certain simplification of the boundary-layer equations. Advantages present themselves

particularly in the consideration of compressible flows (see ref. 1) where g_3 will be chosen in such a manner that the product $\rho(x,y,z) \cdot g_3(x,y,z)$ becomes a suitable function of x and y alone (ρ = Density). For the following considerations, $g_3 = 1$ may be chosen, without loss in generality, so that z then indicates the distance from the wall, measured perpendicularly.

If such a coordinate system is used, the boundary-layer equations for the steady incompressible case read, according to reference 1, as follows:

$$\left.\begin{aligned} &\frac{uu_x}{g_1} + \frac{vu_y}{g_2} + wu_z + uv\frac{g_{1y}}{g_1g_2} - v^2\frac{g_{2x}}{g_1g_2} = -\frac{p_x}{\rho g_1} + \nu u_{zz} \\ &\frac{uv_x}{g_1} + \frac{vv_y}{g_2} + wv_z + uv\frac{g_{2x}}{g_1g_2} - u^2\frac{g_{1y}}{g_1g_2} = -\frac{p_y}{\rho g_2} + \nu v_{zz} \\ &\frac{1}{g_1g_2}\left\{\frac{\partial(g_2u)}{\partial x} + \frac{\partial(g_1v)}{\partial y} + \frac{\partial(g_1g_2w)}{\partial z}\right\} = 0 \end{aligned}\right\} \qquad (2.6)$$

where u, v, w denote the velocity components in the direction of the curvilinear coordinates x, y, z; p is the pressure; and $\nu = \mu/\rho$ is the kinematic viscosity. The boundary conditions

$$\left.\begin{aligned} &u = v = w = 0 \quad \text{for} \quad z = 0 \\ &u \to U \qquad v \to V \quad \text{for} \quad z \to \infty \end{aligned}\right\} \qquad (2.6a)$$

are added, with U and V denoting the components of the velocity of the external flow in the x- and y-directions, respectively. (Initial conditions for unique determination of u and v are disregarded below. Instead, "similar" solutions of equations (2.6) are sought.)

The above equations are based on the following boundary-layer assumptions:

(a) The normal gradients u_z, v_z of the velocity components u, v in the tangential plane of the surface are large compared to the tangential gradients of these velocity components. Corresponding statements apply for the orders of magnitude of the derivatives of u and v.

(b) The normal velocity w is to be treated as negligibly small compared to the tangential velocities; likewise, the corresponding derivatives.

(c) w_z, in contrast, has precisely the same order of magnitude as the tangential derivatives of u and v.

(d) In the boundary layer it may simply be put that $h(x,y,z) = h(x,y,0)$ if h is a function which varies in a comparable manner in all directions (for example, the functions g_{ik}).

(e) Terms affected by the factor ν become noticeable only when they attain, in the sense of assumption (a), the highest order of magnitude, that is, that of u_{zz} and v_{zz}.

Considering the given boundary-layer equations (2.6), it is noticed that of the geometrical data of the surface in the flow, only the metric fundamental quantities enter into the equations. The latter govern, as is well known, the geometry of the surface skin, but not the embedment of the surface in the surrounding space. In the case of equal external pressure distribution, the boundary-layer equations pertaining to isometric surfaces are therefore perfectly identical. This is not too surprising if it is borne in mind that the approximation made with respect to g_{ik} signifies a transfer of the metric conditions to the very thin boundary layer surrounding the surface so that only the inner (metric) geometry of the surface can enter into the whole consideration. Application of this result to the case of a flow around a cylinder yields the well-known statement that effects of the wall curvature on the boundary-layer flow do not appear as long as the assumptions made regarding the g_{ik} prove correct (which is, for instance, no longer the case when the radius of curvature of the wall in the flow is of the order of magnitude of the boundary-layer thickness).

3. DEFINITION OF "SIMILAR" THREE-DIMENSIONAL BOUNDARY LAYERS AND DERIVATION OF THE PERTAINING SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

If a fixed point of the surface in the flow is examined, it is found that there u and v are functions of z alone;

$$u = u(z;x_1,y_1)$$

$$v = v(z;x_1,y_1)$$

These functions of z are also called velocity profiles of the u- or v-component at the point (x_1,y_1).

All solutions of the three-dimensional boundary-layer equations (2.6) will be denoted as "similar" three-dimensional boundary layers for which the u-velocity profiles among them and likewise the v-velocity profiles among them can be made to coincide if u, v, and z are provided with scale factors dependent on the particular point of the wall.

If (x_1,y_1) and (x_2,y_2) are two arbitrary points of the surface in the flow, then

$$\left.\begin{aligned} \frac{u\left(x_1,y_1,\ z/f(x_1,y_1)\right)}{U^*(x_1,y_1)} &= \frac{u\left(x_2,y_2,\ z/f(x_2,y_2)\right)}{U^*(x_2,y_2)} \\ \frac{v\left(x_1,y_1,\ z/f(x_1,y_1)\right)}{V^*(x_1,y_1)} &= \frac{v\left(x_2,y_2,\ z/f(x_2,y_2)\right)}{V^*(x_2,y_2)} \end{aligned}\right\} \qquad (3.1)$$

will therefore apply for "similar" solutions; that is, the functions u/U^* and v/V^* are to be dependent only on $z/f(x,y)$ (f arbitrary dimensionless scale factor for z) or, respectively, on the dimensionless variable

$$\zeta = \frac{z \cdot \sqrt{Re}}{L \cdot f(x,y)} \qquad \text{with} \qquad Re = \frac{U_c \cdot L}{\nu}$$

(U_c = Suitable characteristic velocity, while L = Suitable characteristic dimension of length.[1])

U^* and V^* are scale factors of the dimension of a velocity for u and v. If two velocity components that are different from zero and parallel to the wall are present in the external flow, the pertaining velocity U or V of the external flow must necessarily be chosen as the

---

[1]At this point it is recognized that nothing new would be added if $g_3 \neq 1$ were admitted. If g_3 were any function of x and y: $g_3 = g_3(x,y)$, this would mean only the introduction of another scale factor which is a space function in the z-direction. The product $g_3 \cdot f$ may then be introduced as a new scale factor, which shows that the restriction to $g_3 = 1$ does not involve a loss in generality.

scale factor for u and v. If, on the other hand, no V-component is present in the external flow which corresponds to the use of streamline coordinates,[2] an arbitrary function of the dimension of a velocity as scale factor may be chosen for v. Thus,

$$U^* \equiv U \quad \text{and} \quad V^* \equiv \begin{cases} V \text{ for } V \neq 0 \\ \text{arbitrary scale factor,} \\ \text{if } V \equiv 0 \end{cases} \tag{3.2}$$

Since "similar" three-dimensional boundary layers were defined, with use of the components u and v of the velocity vector in the direction of the curvilinear coordinates on the surface in the flow, it is conceivable that the given concept of similarity depends on the selection of the coordinate system; that is, it is possible that for a given flow about a given surface, with a particular coordinate system being used, the velocity profiles are similar in the indicated sense but are similar no longer if any other coordinate system is chosen. That the latter statement is actually true, is shown by an example given on p. 50. There the use of rectangular Cartesian coordinates leads to "similar" velocity profiles, whereas, when streamline coordinates are used, the obtained velocity profiles are certainly no longer "similar."

The significance of the "similar" velocity profiles explained in this manner lies in the following theorem:

If the profiles of the two velocity components u and v, parallel to the surface in the flow, are "similar," the problem of calculating the boundary-layer flow may be reduced to the solution of a system of two ordinary nonlinear differential equations of the third order with constant coefficients under certain boundary conditions.

In order to show this, two so-called stream functions ψ and φ are first introduced in such a manner that the continuity equation is identically fulfilled.[3] This is achieved by putting:

---

[2]"Streamline coordinates" are understood to be a coordinate grid system on the surface in the flow, one family of coordinate lines of which (x-lines) coincide with the streamlines of the external flow. When such a coordinate system is used, the external flow therefore shows only a single velocity component (U) parallel to the wall which does not disappear.

[3]The introduction of a pair of stream functions for simplification of the three-dimensional boundary-layer equations was carried out for the first time by F. K. Moore in reference 3 for the special case of rectangular Cartesian coordinates. The following proof for the existence of such a pair of stream functions, with use of a vector potential, is a generalization of the proof performed by Moore for the special case of rectangular Cartesian coordinates.

$$\left.\begin{aligned} g_2 u &= \psi_z \\ g_1 v &= \varphi_z \\ g_1 g_2 w &= -\left(\psi_x + \varphi_y\right) \end{aligned}\right\} \tag{3.3}$$

Under certain assumptions of continuity which the flow must satisfy, there always exist two such stream functions. This may be proved as follows:

On the basis of the continuity equation, the velocity vector $\underline{w}$ with the components u, v, w shows a divergence disappearing in the direction of the curvilinear coordinates

$$\operatorname{div} \underline{w} = 0$$

As is known from the vector analysis, any continuously differentiable vector with disappearing divergence may be represented as the curl of a vector potential $\underline{A}$. For the vector $\underline{w}$ there exists therefore always a vector potential $\underline{A}$ with the components A_1, A_2, A_3 which is defined by

$$\underline{w} = \operatorname{curl} \underline{A}$$

In the component notation, this relationship reads

$$g_2 u = \frac{\partial A_3}{\partial y} - \frac{\partial (g_2 A_2)}{\partial z}$$

$$g_1 v = \frac{\partial (g_1 A_1)}{\partial z} - \frac{\partial A_3}{\partial x}$$

$$g_1 g_2 w = \frac{\partial (g_2 A_2)}{\partial x} - \frac{\partial (g_1 A_1)}{\partial y}$$

For every three-dimensional incompressible steady flow with continuously differentiable velocity vector there exists therefore the two functions

$$\psi(x,y,z) = -g_2A_2 + \int_0^z \frac{\partial A_3}{\partial y}\, dz - h(x,y)$$

$$\varphi(x,y,z) = g_1A_1 - \int_0^z \frac{\partial A_3}{\partial x}\, dz + \tilde{h}(x,y)$$

where h and $\tilde{h}$ are two functions which satisfy the condition $h_x + \tilde{h}_y = 0$ and are otherwise arbitrary. Assume that the function $A_3(x,y,z)$ is twice continuously differentiable with respect to x and y. Then

$$\frac{\partial}{\partial x}\int_0^z \frac{\partial A_3}{\partial y}\, dz = \int_0^z \frac{\partial^2 A_3}{\partial y\, \partial x}\, dz = \int_0^z \frac{\partial^2 A_3}{\partial x\, \partial y}\, dz = \frac{\partial}{\partial y}\int_0^z \frac{\partial A_3}{\partial x}\, dz$$

is valid. Thus,

$$\psi_z = -\frac{\partial(g_2A_2)}{\partial z} + \frac{\partial A_3}{\partial y} = g_2u$$

$$\varphi_z = \frac{\partial(g_1A_1)}{\partial z} - \frac{\partial A_3}{\partial x} = g_1v$$

$$-(\psi_x + \varphi_y) = \frac{\partial(g_2A_2)}{\partial x} - \frac{\partial}{\partial x}\int_0^z \frac{\partial A_3}{\partial y}\, dz - h_x -$$

$$\frac{\partial(g_1A_1)}{\partial y} + \frac{\partial}{\partial y}\int_0^z \frac{\partial A_3}{\partial x}\, dz - \tilde{h}_y$$

$$= \frac{\partial(g_2A_2)}{\partial x} - \frac{\partial(g_1A_1)}{\partial y} = g_3g_2w$$

The given functions ψ and φ thus actually represent a pair of stream functions so that the assertion made is proved. On the basis of the arbitrariness of the functions h and $\tilde{h}$ it is recognized that the two stream functions ψ and φ are not unequivocally determined.

Performance of the indicated transformation (3.3) yields from the equations of motion the following system of partial differential equations for ψ and φ:

$$\frac{1}{g_1 g_2^2}\left(\psi_z\psi_{xz} + \varphi_z\psi_{yz} - \psi_x\psi_{zz} - \varphi_y\psi_{zz}\right) - \frac{g_{2x}}{g_1 g_2^3}\psi_z^2 -$$

$$\frac{g_{2x}}{g_1^3 g_2}\varphi_z^2 - \frac{1}{g_1 g_2^2}\left(\frac{g_{2y}}{g_2} - \frac{g_{1y}}{g_1}\right)\varphi_z\psi_z = -\frac{p_x}{\rho g_1} + \frac{\nu}{g_2}\psi_{zzz}$$

$$\frac{1}{g_1^2 g_2}\left(\psi_z\varphi_{xz} + \varphi_z\varphi_{yz} - \psi_x\varphi_{zz} - \varphi_y\varphi_{zz}\right) - \frac{g_{1y}}{g_1^3 g_2}\varphi_z^2 -$$

$$\frac{g_{1y}}{g_1 g_2^3}\psi_z^2 - \frac{1}{g_1^2 g_2}\left(\frac{g_{1x}}{g_1} - \frac{g_{2x}}{g_2}\right)\varphi_z\psi_z = -\frac{p_y}{\rho g_2} + \frac{\nu}{g_1}\varphi_{zzz}$$

From the boundary conditions for the velocity components u, v, w the following boundary conditions for the stream functions result:

$$\psi_z(x,y,0) = \varphi_z(x,y,0) = 0$$

$$\psi_z(x,y,z) \to g_2(x,y) \cdot U(x,y)$$

$$\varphi_z(x,y,z) \to g_1(x,y) \cdot V(x,y) \qquad \text{for } z \to \infty$$

$$\psi_x(x,y,0) + \varphi_y(x,y,0) = 0$$

The boundary condition $\psi_x + \varphi_y = 0$ for $z = 0$, which would generally be hard to manipulate, may be replaced by the following one:

$$\psi(x,y,0) = \varphi(x,y,0) = 0$$

since there always exists at least one pair of stream functions which, additionally, satisfies this boundary condition. In order to understand the latter statement, it is only necessary to choose

$$h(x,y) = g_2(x,y) \cdot A_2(x,y,0) \quad \text{and} \quad \tilde{h}(x,y) = -g_1(x,y) \cdot A_1(x,y,0)$$

Then the two functions ψ and φ disappear for $z = 0$. With the above selection of h and $\tilde{h}$, the condition $h_x + \tilde{h}_y = 0$ is satisfied. Because of $w = 0$ for $z = 0$

$$\left\{\frac{\partial(g_2 A_2)}{\partial x} - \frac{\partial(g_1 A_1)}{\partial y}\right\}_{z=0} = 0$$

is obtained. If the following transformation of variables:

$$\left.\begin{aligned} \xi &= \frac{x}{L} \qquad \eta = \frac{y}{L} \qquad \zeta = \frac{z \cdot \sqrt{Re}}{L \cdot f(x,y)} \\ F(\xi,\eta,\zeta) &= \psi(x,y,z)\frac{\sqrt{Re}}{L \cdot U \cdot f \cdot g_2} \\ G(\xi,\eta,\zeta) &= \varphi(x,y,z)\frac{\sqrt{Re}}{L \cdot V^* \cdot f \cdot g_1} \end{aligned}\right\} \qquad (3.4)$$

is carried out, where primes signify differentiations with respect to ζ,

$$\psi_z = g_2 U F' \qquad \psi_{zz} = g_2 U \frac{\sqrt{Re}}{L \cdot f} \cdot F'' \qquad \nu \cdot \psi_{zzz} = g_2 U \frac{U_c}{Lf^2} F'''$$

$$\psi_x = \frac{g_2 U f}{\sqrt{Re}} F_\xi + \frac{L}{\sqrt{Re}}(g_2 U f)_x \cdot F - \frac{L}{\sqrt{Re}} g_2 U \zeta \frac{f_x}{f} F'$$

$$\psi_{xz} = \frac{g_2 U}{L} F_\xi' + (g_2 U)_x \cdot F' - g_2 J \zeta \frac{f_x}{f} F''$$

$$\psi_{yz} = \frac{g_2 U}{L} F_\eta' + (g_2 U)_y \cdot F' - g_2 J \zeta \frac{f_y}{f} F''$$

$$\varphi_z = g_1 V^* G' \qquad \varphi_{zz} = g_1 V^* \frac{Re}{L \cdot f} G'' \qquad \nu\varphi_{zzz} = g_1 V^* \frac{U_c}{Lf^2} G'''$$

$$\varphi_{xz} = \frac{g_1 V^*}{L} G_\xi' + \left(g_1 V^*\right)_x G' - g_1 V^* \zeta \frac{f_x}{f} \cdot G''$$

$$\varphi_y = \frac{g_1 V^* f}{\sqrt{Re}} \cdot G_\eta + \frac{L}{\sqrt{Re}} \left(g_1 V^* f\right)_y G - \frac{L}{\sqrt{Re}} g_1 V^* \zeta \frac{f_y}{f} \cdot G'$$

$$\varphi_{yz} = \frac{g_1 V^*}{L} G_\eta' + \left(g_1 V^*\right)_y G' - g_1 V^* \zeta \frac{f_y}{f} G''$$

is obtained. Accordingly,

$$F' = \frac{\psi_z}{g_2 U} = \frac{u}{U} \quad \text{and} \quad G' = \frac{\varphi_z}{g_1 V^*} = \frac{v}{V^*}$$

The velocity profiles of u and v are therefore similar in the indicated sense, then and only then, when the functions F' and G' explained according to equations (3.4) depend only on ζ. Hence, it follows that F and G themselves also depend only on ζ; for, quite generally

$$F(\xi,\eta,\zeta) = \int_0^\zeta \frac{\partial F}{\partial \zeta} d\zeta + h_1(\xi,\eta) = J(\xi,\eta,\zeta) + h_1(\xi,\eta)$$

is valid. In the case now considered, where $\frac{\partial F}{\partial \zeta} = F'$ depends only on ζ, J also becomes a function of ζ alone, thus

$$F(\xi,\eta,\zeta) = J(\zeta) + h_1(\xi,\eta)$$

$$J(0) = 0$$

On the basis of the boundary condition $F(\xi,\eta,0) = 0$, however, $h(\xi,\eta)$ must disappear identically, which shows that F itself also can depend only on ζ when $F' = \frac{\partial F}{\partial \zeta}$ is a function of ζ alone. A corresponding type of conclusion applies for G.

Performance of the indicated transformation leads to the following differential equations for F and G:

$$F''' + aFF'' + cGF'' + bF'G' + dF'^2 + eG'^2 + k = \frac{f^2}{U_c}\left\{\frac{U}{g_1} F' \frac{\partial F'}{\partial \xi} + \frac{V^*}{g_2} G' \frac{\partial F'}{\partial \eta} - \frac{U}{g_1} F'' \frac{\partial F}{\partial \xi} - \frac{V^*}{g_2} F'' \frac{\partial G}{\partial \eta}\right\}$$

$$G''' + aFG'' + cGG'' + \bar{b}F'G' + \bar{d}G'^2 + \bar{e}F'^2 + \bar{k} = \frac{f^2}{U_c}\left\{\frac{U}{g_1} F' \frac{\partial G'}{\partial \xi} + \frac{V^*}{g_2} G' \frac{\partial G'}{\partial \eta} - \frac{U}{g_1} G'' \frac{\partial F}{\partial \xi} - \frac{V^*}{g_2} G'' \frac{\partial G}{\partial \eta}\right\}$$

In the case of "similar" solutions now considered F and G depend only on ζ so that the right sides disappear and the following system of two ordinary differential equations of the third order is derived:

$$\left.\begin{aligned} F''' + aFF'' + cGF'' + bF'G' + dF'^2 + eG'^2 + k = 0 \\ G''' + aFG'' + cGG'' + \bar{b}F'G' + \bar{d}G'^2 + \bar{e}F'^2 + \bar{k} = 0 \end{aligned}\right\} \qquad (3.5)$$

with boundary conditions:

$$\left.\begin{aligned} F = F' = G = G' = 0 \\ F' \to 1, \quad G' \to \begin{cases} 1, & \text{if } V \not\equiv 0 \\ 0, & \text{if } V \equiv 0 \end{cases} \qquad \text{for } \zeta \to \infty \end{aligned}\right\} \qquad (3.5a)$$

The coefficients here have the following significance:

$$\left.\begin{aligned} a &= \frac{L}{U_c}\frac{1}{g_1 g_2} f \frac{\partial}{\partial x}\left(g_2 U f\right) & c &= \frac{L}{U_c}\frac{1}{g_1 g_2} f \frac{\partial}{\partial y}\left(g_1 V^* f\right) \\ b &= -\frac{L}{U_c}\frac{1}{g_2} f^2 V^*\left(\frac{U_y}{U} + \frac{g_{1y}}{g_1}\right) & \bar{b} &= -\frac{L}{U_c}\frac{1}{g_1} f^2 U\left(\frac{V^*_x}{V^*} + \frac{g_{2x}}{g_2}\right) \\ d &= -\frac{L}{U_c}\frac{1}{g_1} f^2 U_x & \bar{d} &= -\frac{L}{U_c}\frac{1}{g_2} f^2 V^*_y \\ e &= \frac{L}{U_c}\frac{g_{2x}}{g_1 g_2} f^2 \frac{V^{*2}}{U} & \bar{e} &= \frac{L}{U_c}\frac{g_{1y}}{g_1 g_2} f^2 \frac{U^2}{V^*} \\ k &= -\frac{L}{U_c}\frac{1}{g_1} f^2 \frac{1}{U}\frac{p_x}{\rho} & \bar{k} &= -\frac{L}{U_c}\frac{1}{g_2} f^2 \frac{1}{V^*}\frac{p_y}{\rho} \end{aligned}\right\} \qquad (3.6)$$

The component notation of the hydrodynamic fundamental equation for the frictionless external flow

$$\frac{1}{2}\,\mathrm{grad}\ \underline{w}^2 + \frac{1}{\rho}\,\mathrm{grad}\ p = \left[\underline{w},\ \mathrm{curl}\ \underline{w}\right]$$

yields the following connection between the pressure and the components of the external velocity:

$$\left.\begin{aligned} -\frac{p_x}{\rho} &= UU_x + VV_x + \frac{V}{g_2}\left\{\frac{\partial}{\partial y}\left(g_1 U\right) - \frac{\partial}{\partial x}\left(g_2 V\right)\right\} \\ -\frac{p_y}{\rho} &= UU_y + VV_y - \frac{U}{g_1}\left\{\frac{\partial}{\partial y}\left(g_1 U\right) - \frac{\partial}{\partial x}\left(g_2 V\right)\right\} \end{aligned}\right\} \qquad (3.7)$$

Accordingly, as can be easily checked, in the case $V \not\equiv 0$

$$k = -b - d - e \quad \text{and} \quad \bar{k} = -\bar{b} - \bar{d} - \bar{e} \qquad (3.8)$$

If, in particular, the external flow is free from vortices, there follows from the disappearance of the z-component of curl $\underline{w}$ the relationship

$$\frac{\partial}{\partial x}\left(g_2 V\right) - \frac{\partial}{\partial y}\left(g_1 U\right) = 0 \tag{3.9}$$

Between the coefficients b and $\bar{b}$ there exists then the connection

$$\bar{b} = \frac{U^2}{V^2}\, b \tag{3.10}$$

In the case of streamline coordinates being used

$$-\frac{p_x}{\rho} = UU_x \quad \text{and} \quad -\frac{p_y}{\rho} = -\frac{g_{1y}}{g_1}\, U^2 \tag{3.11}$$

so that for k and $\bar{k}$ the relations

$$k = -d \qquad \bar{k} = -\bar{e} \tag{3.12}$$

are obtained.

The system of equations for F and G may now be written also as follows:

(a) In the case of the presence of two nondisappearing velocity components parallel to the wall in the external flow

$$\left.\begin{aligned} F''' + aFF'' + cGF'' + b(F'G' - 1) + d\left(F'^2 - 1\right) + e\left(G'^2 - 1\right) &= 0 \\ G''' + aFG'' + cGG'' + \bar{b}(F'G' - 1) + \bar{d}\left(G'^2 - 1\right) + \bar{e}\left(F'^2 - 1\right) &= 0 \end{aligned}\right\} \tag{3.13}$$

with boundary conditions: $F = F' = G = G' = 0$ for $\zeta = 0$

$$F' \to 1, \quad G' \to 1 \quad \text{for} \quad \zeta \to \infty$$

(b) In the case of streamline coordinates being used

$$\left.\begin{aligned} F''' + aFF'' + cGF'' + bF'G' + d\left(F'^2 - 1\right) + eG'^2 = 0 \\ G''' + aFG'' + cGG'' + \bar{b}F'G' + \bar{d}G'^2 + \bar{e}\left(F'^2 - 1\right) = 0 \end{aligned}\right\} \tag{3.14}$$

with boundary conditions: $F = F' = G = G' = 0$ for $\zeta = 0$

$$F' \to 1, \quad G' \to 0 \quad \text{for} \quad \zeta \to \infty$$

4. PROOF OF THE CONSTANCY OF THE COEFFICIENTS IN THE DIFFERENTIAL EQUATIONS PERTAINING TO "SIMILAR" SOLUTIONS

At first glance, it is conceivable that the coefficients a, b, c, . . . in the derived differential equations for $F(\zeta)$ and $G(\zeta)$ may be arbitrary functions of x and y; however, this is not the case. Rather, it will be shown in what follows that the boundary-value problem considered has a solution only in the case when the coefficients are constant.[4]

The following proof is based, on the one hand, on the boundary conditions valid at the point $\zeta = 0$ and, on the other, on the study of the linear dependences necessarily existing between the functions of ζ among themselves and the functions of x and y among themselves in the case of nonconstant coefficients.

---

[4]The related problem of the similar solutions of the two-dimensional nonsteady boundary-layer equations shows that boundary-value problems of a similar kind occur where a solution exists for nonconstant coefficients. According to reference 23, the differential equation

$$F''' + a\left(FF'' - F'^2 + 1\right) + b\zeta F'' + cFF'' + d(1 - F') = 0$$

for the function $F(\zeta)$ is led to there. The coefficients a, b, c, d depend on two parameters x and t. Under the boundary conditions $F(0) = F'(0) = 0$, $F'(\infty) = 1$, the differential equation for $a = h(x,t)$, $b = -h(x,t)$, $c = 0$, $d = 1 - h(x,t)$ - where $h(x,t)$ signifies an arbitrary function of x and t - has the following solution which is independent of x and t:

$$F(\zeta) = e^{-\zeta} + \zeta - 1$$

(a) First to be considered is the case in which the choice of coordinates leads to two nonidentically disappearing velocity components parallel to the wall in the external flow. The system of equations for F and G may then be considered in the form

$$\left.\begin{aligned} F''' + aFF'' + cGF'' + bF'G' + dF'^2 + eG'^2 + k = 0 \\ G''' + aFG'' + cGG'' + \bar{b}F'G' + \bar{d}G'^2 + \bar{e}F'^2 + \bar{k} = 0 \end{aligned}\right\} \qquad (4.1)$$

By differentiation with respect to ζ a series of further equations is obtained:

$$\text{I}' \quad F^{IV} + (a + 2d)F'F'' + (b + c)G'F'' + bF'G'' + aFF''' + cGF''' + 2eG'G'' = 0$$

$$\text{II}' \quad G^{IV} + (c + 2\bar{d})G'G'' + (a + \bar{b})F'G'' + \bar{b}G'F'' + cGG''' + aFG''' + 2\bar{e}F'F'' = 0$$

$$\text{I}'' \quad F^{V} + (a + 2d)F''^2 + (2b + c)F''G'' + 2eG''^2 + (2a + 2d)F'F''' +$$
$$(b + 2c)G'F''' + bF'G''' + 2eG'G''' + aFF^{IV} + cGF^{IV} = 0$$

$$\text{II}'' \quad G^{V} + (c + 2\bar{d})G''^2 + (2\bar{b} + a)F''G'' + 2\bar{e}F''^2 + (2c + 2\bar{d})G'G''' +$$
$$(2a + \bar{b})F'G''' + \bar{b}G'F''' + 2\bar{e}F'F''' + cGG^{IV} + aFG^{IV} = 0$$

. . . . . etc. . . . . .

In general, let $\text{I}^{(\nu)}$ or $\text{II}^{(\nu)}$ signify the equation obtained by differentiating the first or second equation of equations (4.1) ν-times. At the point $\zeta = 0$ to which the subscript zero will always refer below, there are obtained, making use of the prescribed boundary conditions, the following equations:

$$\text{I}_0 \quad F_0''' + k = 0$$

$$\text{II}_0 \quad G_0''' + \bar{k} = 0$$

$$\text{I}_0' \quad F_0^{IV} = 0$$

$$\text{II}_0' \quad G_0^{IV} = 0$$

$$\text{I}_0'' \quad F_0^{V} + (a + 2d)F_0''^2 + (2b + c)F_0''G_0'' + 2eG_0''^2 = 0$$

$$\text{II}_0'' \quad G_0^{V} + (c + 2\bar{d})G_0''^2 + (2\bar{b} + a)F_0''G_0'' + 2\bar{e}F_0''^2 = 0$$

. . . . . etc. . . . . .

From I_0 and II_0 it is recognized immediately that

$$k = -(b + d + e) \quad \text{and} \quad \bar{k} = -(\bar{b} + \bar{d} + \bar{e})$$

must be constant.

Let it now be assumed that $F''_0 \neq 0$, $F'''_0 \neq 0$, $F^V_0 \neq 0$. By the following abbreviation:

$$\frac{G''_0}{F''_0} = A \qquad \frac{G'''_0}{F'''_0} = B \qquad \frac{G^V_0}{F^V_0} = C$$

the ensuing equations are set up:

$$① = \frac{I''_0}{F''^2_0} a + 2Ab + Ac + 2d + 2A^2e = C_1$$

$$② = \frac{II''_0}{F''^2_0} Aa + A^2c + 2A\bar{b} + 2A^2\bar{d} + 2\bar{e} = C_2$$

$$③ = \frac{I'''_0}{F''_0F'''_0} 4a + (3A + 3B)b + (3A + B)c + 6d + 6ABe = C_3$$

$$④ = \frac{II'''_0}{F''_0F'''_0}(3B + A)a + 4ABc + (3A + 3B)\bar{b} + 6AB\bar{d} + 6\bar{e} = C_4$$

$$⑤ = \frac{I_0}{F'''^2_0} 4a + 6Bb + 4Bc + 6d + 6B^2e = C_5$$

$$⑥ = \frac{II_0}{F'''^2_0} 4Ba + 4B^2c + 6B\bar{b} + 6B^2\bar{d} + 6\bar{e} = C_6$$

$$⑦ = \frac{I_0}{F''_0F_0} 11a + (5A + 5C)b + (10A + C)c + 10d + 10ACe = C_7$$

$$\textcircled{8} = \frac{II_0}{F_0''F_0}(A + 10C)a + 11ACc + (5A + 5c)\bar{b} + 10AC\bar{d} + 10\bar{e} = C_8$$

$$\textcircled{9} = \textcircled{3} - \textcircled{5}3(A - B) \cdot (b + c + 2Bc) = C_9$$

$$\textcircled{10} = \textcircled{4} - \textcircled{6}(A - B) \cdot (a + 4Bc - 3\bar{b} + 6B\bar{d}) = C_{10}$$

$$\textcircled{11} = 3 \cdot \textcircled{1} - 2 \cdot \textcircled{3} + \textcircled{5} - a + (2B - 3A)c + 6(A - B)^2e = C_{11}$$

$$\textcircled{12} = 3 \cdot \textcircled{2} - 2 \cdot \textcircled{4} + \textcircled{6}(A - 2B)a + 4(A - B)^2 - A^2c + 6(A - B)^2\bar{d} = C_{12}$$

The equations ①, ②, ⑦, ⑧, ⑨, ⑩, ⑪, ⑫ represent a system for calculation of the eight unknowns a, b, c, d, e, $\bar{b}$, $\bar{d}$, $\bar{e}$. If the system determinant $\Delta \neq 0$, unique solution is guaranteed. From the solution formulas the constancy of the coefficients a, b, c, . . . can then be seen directly. Thus, there remains only the case to be considered where Δ disappears. Since

$$\Delta = 3(A - B)^2 \begin{vmatrix} 1 & 2A & A & 2 & 2A^2 & 0 & 0 \\ A & 0 & A^2 & 0 & 0 & 2A^2 & 2 \\ 11 & 5A + 5C & 10A + C & 10 & 10AC & 0 & 0 \\ 10C + A & 0 & 11AC & 0 & 0 & 10AC & 10 \\ 0 & 1 & 1 & 0 & 2B & 0 & 0 \\ 1 & 0 & 4B & 0 & 0 & 6B & 0 \\ -1 & 0 & 2B - 3A & 0 & 6(A - B)^2 & 0 & 0 \\ A - 2B & 0 & 4(A - B)^2 - A^2 & 0 & 0 & 6(A - B)^2 & 0 \end{vmatrix} = -4320(A - B)^6(A - C)^2$$

Δ is then, and only then, equal to zero when A = B or A = C. From equations ③ - 3 · ① or from equations ⑦ - 5 · ① then follows the correctness of the relations

$$\left.\begin{aligned} a + Ac &= \text{Constant} = C_{21} \\ d + Ab + A^2e &= \text{Constant} = C_{22} \\ \bar{e} + A\bar{b} + A^2\bar{d} &= \text{Constant} = C_{23} \end{aligned}\right\} \qquad (4.2)$$

The differential equations (4.1) may then be written also as follows:

$$\left.\begin{aligned} cF''(G - AF) + bF'(G' - AF') + e(G' + AF')(G' - AF') &= H_1 \\ cG''(G - AF) + \bar{b}F'(G' - AF') + \bar{d}(G' + AF')(G' - AF') &= H_2 \end{aligned}\right\} \quad (4.3)$$

where

$$H_1 = -(F''' + k + C_{21}FF'' + C_{22}F'^2)$$

and

$$H_2 = -(G''' + \bar{k} + C_{21}FG'' + C_{23}F'^2)$$

are pure functions of ζ.

Let it now be assumed that $F' \not\equiv \alpha G'$ (α is a constant). Since then the functions of ζ standing on the left side of the above equations cannot all be proportional to one another, a linear dependence must exist between the functions of x and y as well as between the functions of ζ:[5]

$$c = \lambda_1 b + \lambda_2 e \qquad F''(G - AF) = \left\{\mu_1 F' + \mu_2(G' + AF')\right\} \cdot (G' - AF')$$

$$c = \bar{\lambda}_1 \bar{b} + \bar{\lambda}_2 \bar{d} \qquad G''(G - AF) = \left\{\bar{\mu}_1 F' + \bar{\mu}_2(G' + AF')\right\} \cdot (G' - AF')$$

Aside from $G \equiv AF$ this system of equations has no solutions which satisfy the boundary conditions. If $G - AF$ is put equal to D there is obtained the following system of equations for F and D:

---

[5] As can easily be understood, the following auxiliary theorem is valid if in the relation

$$\sum_{\nu=1}^{3} a_\nu(x,y) \cdot A_\nu(\zeta) = H(\zeta)$$

the functions A_1, A_2, A_3, and H are not all proportional to one another and $\not\equiv 0$; the relation can exist only when the a_ν on the one hand and the A_ν on the other are linearly dependent, unless the a_ν are constant.

$$\left.\begin{aligned} DF'' &= \alpha F'D' + \beta D'^2 \\ DD'' + ADF'' &= \bar{\alpha} F'D' + \bar{\beta} D'^2 \end{aligned}\right\} \qquad (4.4)$$

with the boundary conditions

$$F'(0) = 0 \qquad F''(0) = F_0'' \neq 0 \qquad F'''(0) = F_0''' \neq 0 \qquad F'(\infty) = 1$$

$$D(0) = D'(0) = D''(0) = D'''(0) = D(0) = 0 \qquad D'(\infty) = 1 - A$$

If the first equation of (4.3) is multiplied by A and subtracted from the second, the equation

$$DD'' = \gamma F'D' + \delta D'^2 \qquad (\gamma, \ \delta \text{ constant})$$

is obtained, which aside from $D \equiv 0$, does not possess a solution with $D''(0) = 0$, if $\gamma \neq 0$. If $\gamma = 0$, D must satisfy the differential equation

$$DD'' = \delta \cdot D'^2$$

This equation has the solutions

$$D = \left\{\begin{aligned} &\left|C_0 + C_1\zeta\right|^{\frac{1}{1-\delta}} && \text{for} \quad \delta \neq 1 \\ &C_0 e^{C_1\zeta} && \text{for} \quad \delta = 1 \end{aligned}\right\}$$

none of which, aside from $D \equiv 0$, satisfies the boundary conditions $D'(0) = 0$, $D'(\infty) = 1 - A$.

Since, therefore, the indicated linear relations cannot all exist, c, b, e, $\bar{b}$, $\bar{d}$ must be constant. On the basis of the relations (4.2) there then results the constancy of the remaining coefficients.

Only the case $F \equiv \alpha_1 G + \beta_1$ (α_1, β_1 constants) remains to be considered. Because of $G_0'' = AF_0''$, there must exist $\alpha_1 = A$, and because of $F'(\infty) = G'(\infty) = 1$, there is obtained $A = 1$. The first differential equation of equations (4.1) therefore reads in this case:

$$F''' + C_{21}FF'' + C_{22}(F'^2 - 1) = 0$$

The coefficients appearing in it

$$C_{21} = a + c \quad \text{and} \quad C_{22} = b + d + e = \bar{b} + \bar{d} + \bar{e}$$

have already been proved to be constant.

Until now, $F_0'' \neq 0$, $F_0''' \neq 0$, $F_0^V \neq 0$ had been assumed. The treatment of the cases where certain of the values F_0'', F_0''', F_0^V disappear may be carried out in a similar manner with the same result: That the coefficients of the differential equations (4.1) must be constant also in these special cases excluded so far. Since no new viewpoints appear in the argument, the latter will be disregarded here.

(b) In the case where streamline coordinates are used, the system of differential equations for F and G read:

$$\left.\begin{aligned} &\text{(I)} \quad F''' + aFF'' + cGF'' + bF'G' + d(F'^2 - 1) + eG'^2 = 0 \\ &\text{(II)} \quad G''' + aFG'' + cGG'' + \bar{b}F'G' + \bar{d}G'^2 + \bar{e}(F'^2 - 1) = 0 \end{aligned}\right\} \tag{4.5}$$

Boundary conditions: $F(0) = F'(0) = G(0) = G'(0) = 0$

$$F' \to 1, \quad G' \to 0 \quad \text{for} \quad \zeta \to \infty$$

From I_0 and II_0 there can now be seen directly the constancy of d and $\bar{e}$. As in (a), here also the constancy of the coefficients at the point $\zeta = 0$ results from the consideration of the differential equations and their derivatives, if $A \neq B$ and $A \neq C$. For $A = B$ or $A = C$ there are now obtained from the equations ③ - 3 · ① or ⑦ - 5 · ①, with consideration of the constancy of d and $\bar{e}$, the relations

$$\left.\begin{aligned} a + Ac &= \text{Constant} = C_{21} \\ b + Ae &= \text{Constant} = \bar{C}_{22} \\ \bar{b} + A\bar{d} &= \text{Constant} = \bar{C}_{23} \end{aligned}\right\} \tag{4.6}$$

The two last equations are valid only for $A \neq 0$. If, making use of this result, the expression

$$\tilde{D} = \frac{G''}{G'} \cdot (\text{I}) - \frac{F''}{G'} \cdot (\text{II})$$

is formed, the equation

$$eG''(G' - AF') - \bar{d}F''(G' - AF') = \tilde{H}(\zeta)$$

is obtained, or, because of the boundary conditions, certainly $G' \not\equiv AF'$:

$$eG'' - \bar{d}F'' = \frac{\tilde{H}(\zeta)}{G' - AF'} = H^*(\zeta) = \text{Pure function of } \zeta$$

However, in the case of nonconstant e and $\bar{d}$ such a relation can exist only when $G'' \equiv \alpha F''$, which leads to $G' \equiv \tilde{\alpha}F' + \tilde{\beta}$ ($\tilde{\alpha}$, $\tilde{\beta}$ constant). The latter identity, however, cannot be valid, because of the boundary conditions. Therefore e and $\bar{d}$ must be constant. Thus b and $\bar{b}$ also are constant. If all terms which are dependent only on ζ are now put on the right side and divided by F'', which is permissible because of $F'' \not\equiv 0$, the first equation of equations (4.5) takes the following form:

$$aF + cG = H^{**}(\zeta) = \text{Pure function of } \zeta$$

Accordingly, since F and G cannot be proportional, a and c also must be constant.

If $A = 0$, there results, first, from $a - Ac = C_{21}$ the constancy of a. The constancy of $\bar{b}$ results on the basis of the equation $D_0'/F_0''^2$:

$$Ab + A^2e - \bar{b} - A\bar{d} = C_1$$

Now the second equation of equations (4.5) can be written as follows:

$$cGG'' + \bar{d}G'^2 = \text{Pure function of } \zeta$$

However, this relation can exist, for nonconstant c and $\bar{d}$, only in the case where GG'' and G'^2 are proportional

$$GG'' = \lambda G'^2$$

As has already been indicated, this differential equation has no solution, aside from $G \equiv 0$, which satisfies the boundary conditions

$G'(0) = G'(\infty) = 0$ so that c and $\bar{d}$ must necessarily be constant. Under the assumption $G' \not\equiv 0$ the first equation of equations (4.5) can now be expressed in the form

$$bF' + eG' = \text{Pure function of } \zeta$$

Because of $F' \not\equiv \alpha G'$, it can be seen from this directly that b and e also must be constant.

Until now, $G \not\equiv 0$ was assumed. If $G \equiv 0$, the first equation of equations (4.5) reads

$$F''' + aFF'' + d(F'^2 - 1) = 0$$

Because of $d = \text{Constant}$ and $F'' \not\equiv 0$ there follows from this immediately the constancy of a.

A discussion of the cases where certain of the values F''_0, F'''_0, F^V_0 disappear will be ommitted here.

5. SOLUTION OF THE SYSTEM OF CONDITIONAL EQUATIONS (3.6) FOR THE FUNCTIONS U, V^*, g_1, g_2, f

The constancy of the coefficients a, b, c, . . . proved in the preceding section signifies that the defining equations (3.6) given for these coefficients represent a system of partial differential equations which the functions

$$U(x,y), \quad V^*(x,y), \quad g_1(x,y), \quad g_2(x,y), \quad f(x,y)$$

must satisfy if "similar" solutions are to appear. The following section deals with the solution of this system of equations.

First, the following remark must be made: As already noted on p. 8, the concept of "similarity" depends on the choice of the coordinate system on the surface in the flow. A coordinate transformation which merely causes a change in scale of the parameter lines $x = \text{Constant}$, $y = \text{Constant}$, thus a transformation of the shape

$$\left.\begin{aligned} x^* &= \tau_1(x) \\ y^* &= \tau_2(y) \end{aligned}\right\} \qquad (5.1)$$

leaves a possibly existing similarity of the velocity profiles untouched since the resolution into components of the velccity vector remains unchanged. This fact is utilized in what follows.

(a) Solution of the System (3.6) for the Special Case of Rectangular Cartesian Coordinates

In the most general case, the solution of the system of conditional equations (3.6) encounters great difficulties. The solution for the special case

$$g_1 = g_2 = 1$$

(rectangular Cartesian coordinates) will therefore be performed first. With

$$\xi = \frac{x}{L} \qquad \eta = \frac{y}{L} \qquad \bar{U} = \frac{U}{U_c} \qquad \bar{V} = \frac{V^*}{U_c}$$

the system of equations (3.6) then reads

$$a = f \cdot (\bar{U}f)_\xi \tag{5.2.1}$$

$$b = -f^2 \cdot \frac{\bar{V}}{\bar{U}} \cdot \bar{U}_\eta \tag{5.2.2}$$

$$d = -f^2 \cdot \bar{U}_\xi \tag{5.2.3}$$

$$c = f \cdot (\bar{V}f)_\eta \tag{5.2.4}$$

$$\bar{b} = -f^2 \cdot \frac{\bar{U}}{\bar{V}} \cdot \bar{V}_\xi \tag{5.2.5}$$

$$\bar{d} = -f^2 \cdot \bar{V}_\eta \tag{5.2.6}$$

$$e = 0 \qquad \bar{e} = 0$$

Since the equations (5.2.1) and (5.2.4) may also be written

$$a = -d + \frac{1}{2}\,\bar{U}(f^2)_\xi$$

$$c = -\bar{d} + \frac{1}{2}\,\bar{V}(f^2)_\eta$$

there results, first of all

$$(2a + d) = (\bar{U}f^2)_\xi$$

$$(2c + \bar{d}) = (\bar{V}f^2)$$

Thus,

$$\bar{U}f^2 = \alpha\xi + h_1(\eta)$$

$$\bar{V}f^2 = \gamma\eta + h_2(\xi)$$

where $\alpha = 2a + d$ and $\gamma = 2c + \bar{d}$ are constant, and h_1 and h_2 are arbitrary functions of η and ξ. Solution of the two above equations with respect to f^2 yields

$$f^2 = \frac{\alpha\xi + h_1(\eta)}{\bar{U}} = \frac{\gamma\eta + h_2(\xi)}{\bar{V}}$$

that is, between $\bar{U}$ and $\bar{V}$ there exists the connection

$$\bar{V} = \frac{\gamma\eta + h_2(\xi)}{\alpha\xi + h_1(\eta)}\,\bar{U} \tag{5.3}$$

Let it now be assumed that $b \neq 0$, $\bar{b} \neq 0$. From

$$\lambda_1 = \frac{d}{\bar{b}} = \frac{\bar{U}_\xi}{\bar{U}} \cdot \frac{\bar{V}}{\bar{V}_\xi}$$

then follows

$$\frac{\partial}{\partial \xi} \ln \left| \bar{U} \right| = \lambda_1 \frac{\partial}{\partial \xi} \cdot \ln \left| \bar{V} \right|$$

Thus,

$$\ln \left| \bar{U} \right| = \ln \left| \bar{V} \right|^{\lambda_1} + \ln \left| H_1(\eta) \right|$$

Likewise, there results from

$$\frac{1}{\lambda_2} = \frac{\bar{a}}{b} = \frac{\bar{V}_\eta}{\bar{V}} \cdot \frac{\bar{U}}{\bar{U}_\eta}$$

the relation

$$\ln \left| \bar{U} \right| = \ln \left| \bar{V} \right|^{\lambda_2} + \ln \left| H_2(\xi) \right|$$

Thus, there is obtained

$$\ln \left| \bar{V} \right|^{\lambda_1} - \ln \left| \bar{V} \right|^{\lambda_2} = \ln \left| H_2(\xi) \right| - \ln \left| H_1(\eta) \right|$$

and, if $\lambda_1 \neq \lambda_2$

$$\bar{V} = \left\{ H_2(\xi) \right\}^{\frac{1}{\lambda_1 - \lambda_2}} \left\{ H_1(\eta) \right\}^{\frac{1}{\lambda_2 - \lambda_1}}$$

$$\bar{U} = H_2 \cdot \bar{V}^{\lambda_2} = \left\{ H_2(\xi) \right\}^{\frac{\lambda_1}{\lambda_1 - \lambda_2}} \left\{ H_1(\eta) \right\}^{\frac{\lambda_2}{\lambda_2 - \lambda_1}}$$

If this result is substituted into equation (5.3), the following equation is obtained:

$$\left\{ H_2(\xi) \right\}^{\frac{1-\lambda_1}{\lambda_1 - \lambda_2}} \cdot \left\{ H_1(\eta) \right\}^{\frac{\lambda_2 - 1}{\lambda_1 - \lambda_2}} = \left\{ \alpha\xi + h_1(\eta) \right\}^{-1} \cdot \left\{ \gamma\eta + h_2(\xi) \right\}$$

If $\alpha \neq 0$, $\gamma \neq 0$, this relation is certainly satisfied for $h_2(\xi) = \text{Constant} = \delta$, $h_1(\eta) = \text{Constant} = \beta$. Then,

$$H_2(\xi) = (\alpha\xi + \beta)^{\frac{\lambda_1-\lambda_2}{\lambda_1-1}}$$

$$H_1(\eta) = (\gamma\eta + \delta)^{\frac{\lambda_1-\lambda_2}{\lambda_2-1}}$$

By the introduction of new constants with these equations, the following solution of the system (5.2) is obtained:

$$\left.\begin{aligned} \bar{U} &= A(\xi - \xi_0)^m \cdot (\eta - \eta_0)^n \\ \bar{V} &= B(\xi - \xi_0)^{m-1}(\eta - \eta_0)^{n+1} \\ f^2 &= (\xi - \xi_0)^{1-m}(\eta - \eta_0)^{-n} \end{aligned}\right\} \qquad (5.4)$$

For the coefficients a, b, c, . . . there result the values

$$a = \frac{A}{2} \cdot (m + 1) \qquad c = \frac{B}{2} \cdot (n + 2)$$

$$b = Bn \qquad \bar{b} = A \cdot (m - 1)$$

$$d = -Am \qquad \bar{d} = -B \cdot (n + 1)$$

The case $\alpha \neq 0$, $\gamma = 0$ leads to $h_1(\eta) = 0$, $H_1(\eta) = \text{Constant}$, whence follows $\bar{U} = \bar{U}(\eta)$ and thus $b = 0$, in contrast to the assumption $b \neq 0$. Likewise, the case $\alpha = 0$, $\gamma \neq 0$ is not compatible with the assumption $\bar{b} \neq 0$.

If $\alpha = 0$, $\gamma = 0$, there is obtained

$$f^2\bar{U} = h_1(\eta)$$

$$f^2\bar{V} = h_2(\xi)$$

Equations (5.2.2) or equations (5.2.5), respectively, then yield

$$-\frac{b}{h_2(\xi)} = \frac{\bar{U}_\eta}{\bar{U}} = \frac{\partial}{\partial\eta} \ln |\bar{U}|$$

$$-\frac{\bar{b}}{h_1(\eta)} = \frac{V_\xi}{\bar{V}} = \frac{\partial}{\partial\xi} \ln |\bar{V}|$$

Thus,

$$\bar{U} = e^{-\frac{b\eta}{h_2(\xi)} + h_3(\xi)}$$

$$\bar{V} = e^{-\frac{\bar{b}\xi}{h_1(\eta)} + h_4(\eta)}$$

and

$$\frac{\bar{U}}{\bar{V}} = e^{-\frac{b\eta}{h_2(\xi)}} \cdot e^{-\frac{\bar{b}\xi}{h_1(\eta)}} \cdot e^{h_3(\xi)-h_4(\eta)} = h_1(\eta) \cdot h_2^{-1}(\xi)$$

This relation is fulfilled for $h_1(\eta)$ = Constant, $h_2(\xi)$ = Constant. Hence follows $\bar{V} = k\bar{U}$ and $\lambda_1 = \lambda_2 = 1$, which is in contradiction to the assumption $\lambda_1 \neq \lambda_2$.

The case $\lambda_1 = \lambda_2$ leads to

$$\ln |H_1(\eta)| = \ln |H_2(\xi)| = \text{Constant} = A$$

Accordingly, there results

$$\bar{U} = A\bar{V}^{\lambda_1} = A\left\{\frac{\gamma\eta + h_2(\xi)}{\alpha\xi + h_1(\eta)}\right\}^{\lambda_1} \bar{U}^{\lambda_1}$$

and, if $\lambda_1 \neq 1$

$$\bar{U} = A^{\frac{1}{1-\lambda_1}} \left\{ \frac{\gamma\eta + h_2(\xi)}{\alpha\xi + h_1(\eta)} \right\}^{\frac{\lambda_1}{1-\lambda_1}}$$

With use of this result consideration of the relation $\frac{d}{\bar{d}} = \text{Constant}$ (with b, in the case now considered also $\bar{d} \neq 0$) leads to $h_1(\eta) = \text{Constant}$, $h_2(\xi) = \text{Constant}$, so that the following solution of equations (5.2) results:

$$\bar{U} = A\left\{ \frac{\gamma\eta + \delta}{\alpha\xi + \beta} \right\}^m$$

$$\bar{V} = B\left\{ \frac{\gamma\eta + \delta}{\alpha\xi + \beta} \right\}^{m+1}$$

$$f^2 = \frac{(\alpha\xi + \beta)^{m+1}}{(\gamma\eta + \delta)^m}$$

This solution is already contained in solution (5.4).

Next, the case $b \neq 0$, $\bar{b} = 0$ will be considered. From $\bar{b} = 0$ there follows $\bar{V}_\xi = 0$, thus $\bar{V} = \bar{V}(\eta)$ which because of equation (5.2.6), also causes $f^2 = f^2(\eta)$. Thus there must exist $h_2(\xi) = \text{Constant}$; that is,

$$\bar{V}f^2 = (2c + \bar{d})(\eta - \eta_0)$$

If this result is substituted into equation (5.2.6), there is obtained

$$\bar{V} = \bar{A} \cdot (\eta - \eta_0)^\alpha$$

if $(2c + \bar{d}) \neq 0$. Equation (5.2.2) then yields further

$$b = -(2c + \bar{d}) \cdot (\eta - \eta_0)\frac{\bar{U}_\eta}{\bar{U}}$$

whence follows

$$\bar{U} = B(\xi) \cdot (\eta - \eta_0)^{\beta}$$

and

$$d = -(2c + \bar{d})\frac{1}{A}(\eta - \eta_0)^{1-\alpha+\beta} \cdot B'(\xi)$$

This relation can exist with constant d only when

$$B(\xi) = \text{Constant}$$

or

$$B'(\xi) = \text{Constant}$$

Thus,

$$B(\xi) = \tilde{D}(\xi - \xi_0) \quad \text{and} \quad \beta = c - 1$$

By the introduction of new constants, the following solutions of equations (5.2) are obtained:

$$\left.\begin{aligned} \bar{U} &= A(\eta - \eta_0)^m \\ \bar{V} &= B(\eta - \eta_0)^n \\ f^2 &= (\eta - \eta_0)^{1-n} \end{aligned}\right\} \qquad (5.5)$$

where the pertaining values of the coefficients are:

$$a = 0 \qquad c = \frac{B}{2} \cdot (n + 1)$$

$$b = -mB \qquad \bar{b} = 0$$

$$d = 0 \qquad \bar{d} = -nB$$

$$\bar{U} = (A\xi + D)(\eta - \eta_0)^m \qquad \bar{V} = B(\eta - \eta_0)^{m+1} \qquad f^2 = (\eta - \eta_0)^{-m}$$

This latter solution is already contained in equations (5.4). For $(2c + \bar{d}) = 0$, quite correspondingly, the following solutions are obtained:

$$\left.\begin{aligned} \bar{U} &= A \cdot e^{\alpha\eta} \\ \bar{V} &= B \cdot e^{\beta\eta} \\ f^2 &= e^{-\beta\eta} \end{aligned}\right\} \tag{5.6}$$

where the pertaining values of the coefficients are:

$$a = 0 \qquad c = \frac{1}{2} \cdot B\beta$$

$$b = -B\alpha \qquad \bar{b} = 0$$

$$d = 0 \qquad \bar{d} = -B\beta$$

$$\left.\begin{aligned} \bar{U} &= (A\xi + D) \cdot e^{\alpha\eta} \\ \bar{V} &= B \cdot e^{\alpha\eta} \\ f^2 &= e^{-\alpha\eta} \end{aligned}\right\} \tag{5.7}$$

where the pertaining values of the coefficients are:

$$a = A \qquad c = \frac{1}{2} \cdot B\alpha$$

$$b = -B\alpha \qquad \bar{b} = 0$$

$$d = -A \qquad \bar{d} = -B\alpha$$

If $b = 0$, $\bar{b} \neq 0$, the following solutions result in complete analogy:

$$\left.\begin{aligned} \bar{U} &= A(\xi - \xi_0)^m \\ \bar{V} &= B(\xi - \xi_0)^n \\ f^2 &= (\xi - \xi_0)^{1-m} \end{aligned}\right\} \tag{5.8}$$

where the pertaining values of the coefficients are:

$$a = \frac{A}{2}(m + 1) \qquad c = 0$$

$$b = 0 \qquad \bar{b} = -nA$$

$$d = -mA \qquad \bar{d} = 0$$

$$\left.\begin{aligned} \bar{U} &= A \cdot e^{\alpha\xi} \\ \bar{V} &= B \cdot e^{\beta\xi} \\ f^2 &= e^{-\alpha\xi} \end{aligned}\right\} \tag{5.9}$$

where the pertaining values of the coefficients are:

$$a = \frac{1}{2} A\alpha \qquad c = 0$$

$$b = 0 \qquad \bar{b} = -A\beta$$

$$d = -A\alpha \qquad \bar{d} = 0$$

$$\left.\begin{aligned} \bar{U} &= A \cdot e^{\alpha\xi} \\ \bar{V} &= (B\eta + D) \cdot e^{\alpha\xi} \\ f^2 &= e^{-\alpha\xi} \end{aligned}\right\} \tag{5.10}$$

where the pertaining values of the coefficients are:

$$a = \frac{1}{2} A\alpha \qquad c = B$$

$$b = 0 \qquad \bar{b} = -A\alpha$$

$$d = -A\alpha \qquad \bar{d} = -B$$

Finally, the case in which b and $\bar{b}$ both disappear must be considered. From equation (5.2.2) or equation (5.2.5) then follows immediately:

$$\bar{U} = \bar{U}(\xi) \qquad \bar{V} = \bar{V}(\eta)$$

Equation (5.2.3) or equation (5.2.6) then yields, if $d \neq 0$, $\bar{d} \neq 0$

$$f^2 = f^2(\xi) \quad \text{or} \quad f^2 = f^2(\eta)$$

that is, there can only be $f^2 = \text{Constant}$; thus, there is obtained the solution

$$\bar{U} = A\xi + B \qquad \bar{V} = C\eta + D \qquad f^2 = 1$$

which is already contained in equations (5.4). If d or $\bar{d}$, or d and $\bar{d}$ disappear, solutions which are already contained in equations (5.5), (5.6), (5.8), or (5.9) are obtained.

(b) Solution of the System (3.6) With Use of the Relation (5.11)

Admitting an arbitrary orthogonal coordinate system, the solution of the system (3.6) will now be performed for the case where the relation

$$(g_1 U)_y - (g_2 V^*)_x = 0 \tag{5.11}$$

exists between U and V^*. If two velocity components different from zero and parallel to the wall are present in the external flow, equation (5.11) coincides with equation (3.9); that is, equation (5.11) then signifies particularly the freedom from vortices of the external flow.

If equation (5.11) is valid, the connection between b and $\bar{b}$ is as follows:

$$\bar{b} = \frac{\bar{U}^2}{\bar{V}^2} b \tag{5.12}$$

On the basis of the proved constancy of the coefficients, it is recognized directly from this that the quotient $\bar{U}/\bar{V}$ is different from a constant only in the case when $b = \bar{b} = 0$. This case will be considered first.

From $b = \bar{b} = 0$ follows:

$$\frac{\bar{U}_\eta}{\bar{U}} + \frac{g_{1\eta}}{g_1} = 0 \quad \text{and} \quad \frac{\bar{V}_\xi}{\bar{V}} + \frac{g_{2\xi}}{g_2} = 0$$

Thus,

$$\frac{\partial}{\partial\eta} \ln \left|\bar{U}g_1\right| = 0 \qquad \frac{\partial}{\partial\xi} \ln \left|\bar{V}g_2\right| = 0$$

that is, $\bar{U}g_1$ is a pure function of ξ and $\bar{V}g_2$ is a pure function of η; thus,

$$g_1 = \frac{H(\xi)}{\bar{U}} \qquad g_2 = \frac{H^*(\tau)}{\bar{V}}$$

According to the remarks made at the beginning of this section, staying within the limits of $H(\xi) = H^*(\eta) = 1$ does not involve a loss of generality, since this case may always be attained by a transformation of the shape (5.1). The system (3.6) therefore reads in the case now considered:

$$\left.\begin{aligned}
a &= \bar{U} \cdot \bar{V} \cdot f \cdot \left(\frac{\bar{U}}{\bar{V}} f\right)_\xi = -d + \frac{\bar{U}^2}{\bar{V}} \cdot e + \frac{\bar{U}^2}{2}(f^2)_\xi \\
c &= \bar{U} \cdot \bar{V} \cdot f \cdot \left(\frac{\bar{V}}{\bar{U}} f\right)_\eta = -\bar{d} + \frac{\bar{V}^2}{\bar{U}^2} \cdot \bar{e} + \frac{\bar{V}^2}{2}(f^2)_\eta \\
b &= 0 \\
\bar{b} &= 0 \\
d &= -\frac{1}{2} f^2\left(\bar{U}^2\right)_\xi \\
\bar{d} &= -\frac{1}{2} f^2\left(\bar{V}^2\right)_\eta \\
e &= -\frac{1}{2} f^2\left(\bar{V}^2\right)_\xi \\
\bar{e} &= -\frac{1}{2} f^2\left(\bar{U}^2\right)_\eta
\end{aligned}\right\} \qquad (5.13)$$

Given here, without the completely exact calculation that is obtained in the individual cases, in addition to certain solutions already enumerated in section (a), are the following solutions of this system of equations (under the assumption that all expressions given below do exist):

$$\left.\begin{aligned} \bar{U}^2 &= A\xi + B\eta + C \\ \bar{V}^2 &= k\bar{U}^2 \\ f^2 &= 1 \\ g_1 &= \frac{1}{\bar{U}} \\ g_2 &= \frac{1}{\bar{V}} \end{aligned}\right\} \tag{5.14}$$

where the pertaining values of the coefficients are:

$$a = 0 \qquad c = 0$$

$$d = -A \qquad \bar{d} = -kB$$

$$e = -kA \qquad \bar{e} = -B$$

$$\left.\begin{aligned} \bar{U} &= 1 \\ \bar{V} &= \frac{A}{\sqrt{1 - \varphi_1'}} \\ f^2 &= 2\varphi_1 \\ g_1 &= 1 \\ g_2 &= \frac{\sqrt{1 - \varphi_1'}}{A} \end{aligned}\right\} \tag{5.15}$$

where $\varphi_1 = \varphi_1(\xi)$ is to be a solution of the differential equation $\varphi_1\varphi_1'' + (\varphi_1' - 1)^2 = 0$ different from (ξ + Constant), where the pertaining values of the coefficients are:

$$a = 1 \qquad c = 0$$

$$d = 0 \qquad \bar{d} = 0$$

$$e = A^2 \qquad \bar{e} = 0$$

$$\left.\begin{aligned} \bar{U} &= \varphi_2(\xi) \\ \bar{V} &= B\sqrt{\bar{U}^2 + A} \\ f^2 &= -\frac{1}{\varphi_2'(\xi)} \\ g_1 &= 1 \\ g_2 &= \frac{1}{B\sqrt{\bar{U}^2 + A}} \end{aligned}\right\} \qquad (5.16)$$

where $\varphi_2 = \varphi_2(\xi)$ is to be a nonconstant solution of the differential equation

$$(\varphi_2^2 + A)\varphi_2'' + \left(\frac{1}{2}\varphi_2^3 - C\varphi_2^2 + \frac{1}{2}A\varphi_2 \cdot\cdot AC\right)\varphi_2'^2 = 0$$

where the pertaining values of the coefficients are:

$$a = C \qquad c = 0$$

$$d = 1 \qquad \bar{d} = 0$$

$$e = B \qquad \bar{e} = 0$$

Aside from $b = \bar{b} = 0$ there remains the case to be considered where $\bar{U}$ and $\bar{V}$ are proportional:

$$\bar{V} = k \cdot \bar{U}$$

The system (3.6) then reads

$$\left.\begin{aligned} a &= -d + \frac{e}{k^2} + \frac{\bar{U}}{2g_1}(f^2)_\xi \\ c &= -\bar{d} + k^2\bar{e} + \frac{k\bar{U}}{2g_2}(f^2)_\eta \\ d &= -\frac{1}{g_1} f^2\bar{U}_\xi \\ \bar{d} &= -k\frac{1}{g_2} f^2\bar{U}_\eta \\ e &= k^2 \frac{g_{2\xi}}{g_1 g_2} f^2\bar{U} \end{aligned}\right\} \qquad (5.17)$$

Because of the relation following from equation (5.11)

$$k(g_2\bar{U})_\xi = (g_1\bar{U})_\eta$$

there applies further:

$$\bar{e} = \frac{1}{k^2}(e + \bar{d}) - d$$

$$b = k^2 d - e$$

$$\bar{b} = \frac{1}{k^2} b$$

In the individual cases, the following solutions of equations (5.17) are obtained:

$$\left.\begin{aligned} \bar{U} &= A\xi^m \\ \bar{V} &= k\bar{U} \\ f^2 &= \xi^{1-m} \\ g_1 &= 1 \\ g_2 &= \xi^n \end{aligned}\right\} \qquad (n = -m) \qquad (5.18)$$

where the pertaining values of the coefficients are:

$$a = +\frac{A}{2}(m + 2n + 1) \qquad c = 0$$

$$b = 0 \qquad \bar{b} = -A(m + n)$$

$$d = -Am \qquad \bar{d} = 0$$

$$e = Ak^2n \qquad \bar{e} = 0$$

$$\left.\begin{aligned} \bar{U} &= A\xi^m \cdot e^{\alpha\eta} \\ \bar{V} &= k\bar{U} \\ f^2 &= \xi^{1-m} \cdot e^{-\alpha\eta} \\ g_1 &= 1 \\ g_2 &= \xi \end{aligned}\right\} \qquad (\alpha = k(1 + m)) \qquad (5.19)$$

where the pertaining values of the coefficients are:

$$a = -\frac{A}{2}(m + 3) \qquad c = \frac{1}{2} \cdot k\alpha$$

$$b = -kA\alpha \qquad \bar{b} = -A(m - 1)$$

$$d = -Am \qquad \bar{d} = -kA\alpha$$

$$e = k^2A \qquad \bar{e} = 0$$

$$\left.\begin{aligned} \bar{U} &= A \cdot \xi^n \cdot \eta^m \\ \bar{V} &= k\bar{U} \\ f^2 &= \xi^{1-n}\eta^{1-m} \\ g_1 &= \alpha\eta \\ g_2 &= \beta\xi \end{aligned}\right\} \qquad \left(\beta = \frac{1+m}{1+n}\frac{\alpha}{k}\right) \qquad (5.20)$$

where the pertaining values of the coefficients are:

$$a = \frac{A}{2\alpha}(3+n) \qquad c = \frac{kA}{2}(3+m)$$

$$b = -\frac{kA}{\beta}(m+1) \qquad \bar{b} = -\frac{A}{\alpha}(n+1)$$

$$d = -\frac{A}{\alpha}n \qquad \bar{d} = -\frac{kA}{\beta} \cdot m$$

$$e = \frac{k^2A}{\alpha} \qquad \bar{e} = \frac{A}{k\beta}$$

$$\left.\begin{aligned} \bar{U} &= 1 \\ \bar{V} &= k \\ f^2 &= (\xi + \alpha\eta)^{m+1} \\ g_1 = g_2 &= (\xi + \alpha\eta)^m \end{aligned}\right\} \qquad (\alpha = k) \qquad (5.21)$$

where the pertaining values of the coefficients are:

$$a = \frac{1}{2} \cdot (3m+1) \qquad c = \frac{k}{2}\alpha(3m+1)$$

$$b = -k\alpha m \qquad \bar{b} = -m$$

$$d = 0 \qquad \bar{d} = 0$$

$$e = mk^2 \qquad \bar{e} = \frac{\alpha}{k}m$$

$$\left.\begin{aligned} \bar{U} &= A \cdot e^{\alpha\xi} \\ \bar{V} &= k\bar{U} \\ f^2 &= e^{-\alpha\xi} \\ g_1 &= 1 \\ g_2 &= e^{\beta\xi} \end{aligned}\right\} \qquad (\beta = -\alpha) \qquad (5.22)$$

where the pertaining values of the coefficients are:

$$a = \frac{A}{2}(\alpha + 2\beta) \qquad c = 0$$

$$b = 0 \qquad \bar{b} = -A(\alpha + \beta)$$

$$d = -A\alpha \qquad \bar{d} = 0$$

$$e = k^2 A\beta \qquad \bar{e} = 0$$

$$\left.\begin{aligned} \bar{U} &= 1 \\ \bar{V} &= k \\ f^2 &= 1 \\ g_1 &= \frac{g_{2\xi}}{g_2} \\ g_2 &= \varphi_3(\xi,\eta) \end{aligned}\right\} \tag{5.23}$$

where $\varphi_3(\xi,\eta)$ is to be a solution of the differential equation

$$\varphi_3\varphi_{3\xi\eta} - \varphi_{3\xi}\varphi_{3\eta} - k\varphi_3{}^2\varphi_{3\xi} = 0$$

where the pertaining values of the coefficients are:

$$a = 1 \qquad c = k^2$$

$$b = -k^2 \qquad \bar{b} = 1$$

$$d = 0 \qquad \bar{d} = 0$$

$$e = k^2 \qquad \bar{e} = 1$$

$$\left.\begin{aligned} &\bar{U} = \varphi_4(\xi,\eta) \qquad \bar{V} = k\bar{U} \\ &f^2 = \frac{1}{\varphi_4(\xi,\eta)} \qquad g_1 = \frac{\varphi_{4\xi}}{\varphi_4} \qquad g_2 = A\frac{\varphi_{4\eta}}{\varphi_4} \end{aligned}\right\} \quad \left(A = \frac{1}{k}\right) \tag{5.24}$$

where $\varphi_4(\xi,\eta)$ is to be a solution of the differential equation

$$\varphi_4\varphi_{4\xi\eta} + C\varphi_{4\xi}\varphi_{4\eta} = 0$$

where the pertaining values of the coefficients are:

$$a = -\left(\frac{1}{2} + C\right) \qquad c = -\frac{k}{A} \cdot \left(\frac{1}{2} + C\right)$$

$$b = C \cdot \frac{k}{A} \qquad \bar{b} = C$$

$$d = -1 \qquad \bar{d} = -\frac{k}{A}$$

$$e = -k^2(1 + C) \qquad \bar{e} = -\frac{1}{Ak}(1 + C)$$

As may be easily checked, the given solutions still satisfy the full system of equations (3.6) if the relations indicated in brackets are not satisfied. However, the external flow is then no longer free from vortices.

On the basis of the symmetry of the system of equations (3.6), a series of further solutions is obtained, if in the given solutions, ξ and η, $\bar{U}$ and $\bar{V}$, and g_1 and g_2 are interchanged. In order to obtain the pertaining values of the coefficients a, b, c, . . ., the following interchange has to be performed:

$$a \leftrightarrow c \qquad (b,d,e) \leftrightarrow (\bar{b},\bar{d},\bar{e})$$

Regarding the indicated values of the coefficients a, b, . . . of the differential equations (3.5), the following remark is made: As can be seen from equations (3.6), a common positive factor of a, c, b, d, e, $\bar{b}$, $\bar{d}$, $\bar{e}$ signifies only a change in scale of f. Since such a change does not matter here, a coefficient different from zero may be normalized to ± 1 in every single case.

6. ON THE INTERPRETATION OF THE CALCULATED SOLUTIONS OF THE CONDITIONAL EQUATIONS (3.6)

In the conditional equations (3.6) there appeared, besides U and V*, also the functions g_1 and g_2; this means that not only the external flow, but also the arc-element square on the surface in the flow, must show a particular shape if "similar" solutions are to occur. Since by the functions g_1 and g_2 merely the metric relations of a surface are determined, but not its shape in the surrounding space, the surface skin in the flow is defined by the solutions calculated in the preceding section only up to bending deformations in every given case.

According to the definition of V^* in the identities (3.2), two cases have to be discussed for the interpretation of U and V^*: Either U and V^* are to be interpreted as velocity components of the external flow in the direction of the curvilinear coordinates on the surface in the flow (the external streamlines then do not coincide with either of the two families of coordinate lines on the surface); or U may be interpreted as the only velocity component present in the external flow, so that the external streamlines coincide with the one family of parameter lines, namely the x-lines, on the surface in the flow. V^* is then to be interpreted as a scale factor for the secondary flow component v appearing in the boundary layer, affected by the dimension of a velocity which has no relation to the external flow.

After these general preliminary remarks, a few of the solutions found in the preceding section will be considered more closely.

(a) The "Similar" Solutions of the Two-Dimensional Problem

The solutions given in section 5 under equations (5.8) and (5.9) lead, with use of streamline coordinates, to the well-known two-dimensional case. If one chooses $n = 0$ or $\beta = 0$, the pertaining differential equations read:

$$\left.\begin{aligned} &F''' + aFF'' + d\left(F'^2 - 1\right) = 0 \\ &G''' + aFG'' = 0 \end{aligned}\right\} \qquad (6.1)$$

with boundary conditions: $F(0) = F'(0) = 0 \qquad F'(\infty) = 1$

$$G'(0) = 0 \qquad G'(\infty) = 0$$

The second equation may be solved

$$G'' = \alpha e^{-\int_0^\zeta F(t)dt} \neq 0$$

according to the boundary-value problem for G which has only the trivial solution $G \equiv 0$. The first equation $F''' + aFF'' + d(F'^2 - 1) = 0$ with $a = \frac{A}{2} \cdot (m + 1)$, $d = -Am$ $\left(\text{or, respectively, } \varepsilon = \frac{A}{2}\alpha \text{ and } d = -A\alpha\right)$ is the well-known differential equation of the "similar" solutions of the two-dimensional problem. According to equations (5.8) and (5.9) the pertaining external velocity distributions are

$$\bar{U} = A\xi^m \quad \text{or} \quad \bar{U} = A \cdot e^{\alpha\xi}$$

Their physical interpretation was carried out completely by Mangler in reference 19. The potential flow about a wedge of the opening angle $\beta\pi = \frac{2m}{1+m}\pi$, which leads to the velocity distribution $\bar{U} = A\xi^m$ with $m > 0$, is particularly of practical significance here.

(b) Corresponding "Similar" Solutions in the Case of a Body of Revolution in an Axially Symmetrical Flow

If for a cone of revolution, the generating-surface lines and the parallel circles are selected as parameter lines, there is obtained $g_1 = 1$, $g_2 = \xi$. According to a remark of Mangler in reference 5, the external flow shows, in the case of an axially symmetrical approach flow to a cone of revolution, the velocity distribution $\bar{U} = A\xi^{\tilde{m}}$ where the exponent $\tilde{m} > 0$ is a function of the opening angle ϑ_0 of the cone of revolution. The solution given under equations (5.18) may therefore be interpreted for $n = -1$ (if $\bar{U}$ is regarded as the only velocity component present in the external flow) as an axially symmetrical flow about a cone of revolution. The pertaining differential equations read

$$\left.\begin{aligned} F''' + aFF'' + d\left(F'^2 - 1\right) + eG'^2 = 0 \\ G''' + aFG'' + \bar{b}F'G' = 0 \end{aligned}\right\} \tag{6.2}$$

with boundary conditions: $F(0) = F'(0) = 0 \qquad F'(\infty) = 1$

$$G(0) = G'(0) = 0 \qquad G'(\infty) = 0$$

Here, $a > 0$, $\bar{b} < 0$. As can easily be understood, by considering the geometrical facts, the second equation has, under the additional assumption $F' > 0$ in combination with the boundary conditions for G, only the trivial solution $G \equiv 0$. (The assumption of a maximum or minimum of G' would lead to a contradiction in sign in the second equation of (6.2).) For F' there remains therefore the differential equation

$$F''' + aFF'' + d\left(F'^2 - 1\right) = 0 \tag{6.2a}$$

a and d are connected with the constants of the external velocity distribution as follows:

$$a = \frac{A}{2}(\tilde{m} + 3) \qquad d = -A\tilde{m}$$

or, with introduction of a new scale factor $f^{+2} = \frac{1}{3} \cdot f^2$

$$a = \frac{A}{2}\left(\frac{\tilde{m}}{3} + 1\right) \qquad d = -A\frac{\tilde{m}}{3}$$

For $m = \frac{\tilde{m}}{3}$, the differential equation occurring for the flow about a wedge accordingly agrees completely with the above differential equation (6.2a); that is, the calculation of the boundary layer on a cone of revolution in an axially symmetrical approach flow with the external velocity distribution $\overline{U} = A\xi^{\tilde{m}}$ coincides with the calculation of the boundary layer on a wedge with the opening angle $\beta\pi = \frac{2m}{1 + m}\pi$ with $m = \frac{\tilde{m}}{3}$. For $m = 1$ to which pertains the opening angle $\vartheta_o = \pi$ of the cone of revolution, there is obtained in particular, the statement that the calculation of the rotationally symmetrical stagnation-point flow can be reduced to that of the two-dimensional wedge flow with the wedge angle $\pi/2$. Mangler obtained the same result in reference 5 with use of "Mangler's transformation."

(c) Line Source and Sink, Three-Dimensional

Source and Sink Flow

If the grid of parameter lines for $n = -1$ in equations (5.18) is interpreted as plane polar coordinates, and $\overline{U}$ again as the only velocity component present in the external flow, the case $m = -1$ corresponds, for $A > 0$, to a line source (or, for $A < 0$, to a line sink) perpendicularly to the wall, since then the external flow shows the velocity distribution $\overline{U} = \frac{A}{\xi}$.

Correspondingly, the case $m = -2$ occurs in the case of the presence of a three-dimensional source or sink at the zero point. With limitation to the only physically meaningful solution $G \equiv 0$ for the secondary flow, there is obtained for F again the differential equation

$$F''' + aFF'' + d\left(F'^2 - 1\right) = 0 \qquad (6.3)$$

with $a = d = A$ in the case of the line source ($A > 0$) or line sink ($A < 0$), and with $a = \frac{A}{2}$, $d = 2A$ in the case of the three-dimensional source ($A > 0$) or sink ($A < 0$).

(d) Rotary Flows About Bodies of Revolution and Flows About Bodies of Revolution Rotating in a Fluid Otherwise at Rest

If on a body of revolution the coordinate system is chosen so that the ξ-lines coincide with the parallel circles and the η-lines with the meridian curves, the arc-element square reads

$$d\bar{s}^2 = \bar{r}^2(\eta)d\xi^2 + d\eta^2$$

when η is the arc length of a meridian curve and $\bar{r} = \bar{r}(\eta)$ denotes the distance from the axis of rotation, made dimensionless with L. Accordingly, the solution

$$\bar{U} = A\eta^m \qquad g_1 = \eta^n \qquad g_2 = 1$$

originating from the solution presented in equations (5.18) by interchange of ξ and η, $\bar{U}$ and $\bar{V}$, g_1 and g_2 may be interpreted as rotary flow about a body of revolution the meridian curve of which satisfies the equation $\bar{r}(\eta) = \eta^n$. The axis of the rotary flow coincides with the axis of the body of revolution. The pertaining system of equations reads:

$$\left.\begin{aligned} F''' + cGF'' + bF'G' &= 0 \\ G''' + cGG'' + \bar{d}G'^2 + \bar{e}\left(F'^2 - 1\right) &= 0 \end{aligned}\right\} \tag{6.4}$$

with boundary conditions: $F'(0) = 0$ $\qquad F'(\infty) = 1$

$G(0) = G'(0) = 0$ $\qquad G'(\infty) = 0$

where

$$c = \frac{A}{2} \cdot (2n + m + 1) \qquad b = -A(m + n)$$

$$\bar{d} = -Am \qquad \bar{e} = An$$

If, in particular, the fluid rotates as a rigid body, there is obtained $U = \omega r$ (ω = Angular velocity); thus, $m = n$. The case $n = 1$ leads to rotary flows about cones of revolution especially to rotary flows over plane surfaces, with the rotational velocity proportional to a power of the distance from the axis of rotation.

If $m = -1$, a potential vortex exists; whereas, for $m = n = 1$ the fluid rotates as a rigid body over a plane. The latter case was treated by U. T. Bödewadt as an exact solution of the complete Navier-Stokes equations in reference 6. The system occurring in the Bödewadt reference, equivalent to the above system (6.4) of two differential equations of the second order and one differential equation of the first order, was solved in a very laborious manner by combination of a power-series expansion about $\zeta = 0$ and of an asymptotical expansion for a mean value of ζ. A nonmonotonic solution for both velocity components resulted; the component in the direction of the main flow showed a considerable excess velocity.

It may be understood in what follows that the system of equations (6.4) has quite generally for $m > 0$, $n > 0$ only solutions in which the primary flow component F' shows an excess velocity ($F' > 1$):

From $m > 0$, $n > 0$ follows $c > 0$, $b < 0$, $\bar{d} < 0$, $\bar{e} > 0$ so that the system of equations (6.4) can also be written as follows:

$$F''' + |c|GF'' - |b|F'G' = 0$$

$$G''' + |c|GG'' - |\bar{d}|G'^2 + |\bar{e}|\left(F'^2 - 1\right) = 0$$

If F' always were <1, thus $\left(F'^2 - 1\right) < 0$, G' could not show a maximum since then a contradiction in sign would occur in the second equation. On the basis of the boundary conditions $G'(0) = G'(\infty) = 0$, G' can then only be always negative. In order for the boundary condition at infinity to be satisfied, there must be, from a certain point onward, for geometrical reasons, $G'' > 0$ and $G''' < 0$ which, because of $G < 0$, again leads to a contradiction in sign in the second equation. Thus only solutions can exist for which F' becomes at some time $F' > 1$.

If the boundary-layer flow which develops when a body of revolution rotates about its axis in a fluid otherwise at rest is considered, instead

of the rotary flow with $U = \omega r$ $(m = n)$ about a body of revolution, the pertaining boundary-layer equations differ from the case of a rotary flow considered until now only in the fact that the pressure term disappears and the boundary conditions for the primary flow component at the wall and at the outer edge of the boundary layer (or at infinity) are exactly interchanged. For bodies of revolution of the indicated shape, the boundary-layer flow therefore shows "similar" velocity profiles also in the latter case; the scale factor for the two components u and v of the boundary-layer flow now is the surface velocity $U = \omega r$ of the rotating body of revolution. Because of $m = n$ and the disappearance of the pressure term the pertaining differential equations, when U is made dimensionless with ωL so that $\bar{U}$ becomes $\bar{U} = \bar{r}$, now read:

$$F''' + \frac{3m + 1}{2} GF'' - 2mF'G' = 0$$

$$G''' + \frac{3m + 1}{2} GG'' - mG'^2 + mF'^2 = 0$$

By limitation to the case $m > 0$ and introduction of a new scale factor $f^{*2} = \frac{1}{m} f^2$, the above equations may now also be written as follows:

$$\left.\begin{aligned} F''' + cGF'' - 2F'G' &= 0 \\ G''' + cGG'' - G'^2 + F'^2 &= 0 \end{aligned}\right\} \quad \text{with} \quad c = \frac{3m + 1}{2m} > 0 \qquad (6.5)$$

with boundary conditions: $F'(0) = 1$ $\qquad F'(\infty) = 0$

$$G(0) = G'(0) = 0 \qquad G'(\infty) = 0$$

The case $m = 1$ $(c = 2)$ which corresponds to the flow near a disk rotating in a fluid otherwise at rest has already been treated in detail. (See refs. 7, 21, and 22.)

"Similar" solutions occur in the case of axially symmetrical approach flow as well as of the rotation of a cone of revolution. This suggests the question whether "similar" solutions are not obtained also in the case of cones of revolution which are in an axially symmetrical approach flow and at the same time rotate about their axes. Since in this case it is necessary to use as a scale factor for the primary flow component u the velocity U of the external flow, and for the component v the surface velocity V of the rotating cone of revolution, and since, according to equations (5.18), "similar" solutions occur only

when V is proportional to U, there is obtained only for $\overline{U} = A\xi$ a "similar" three-dimensional boundary layer. This external velocity distribution appears when the opening angle of the cone is π, that is, in the case of a two-dimensional disk in an axially symmetrical approach flow. Thus, "similar" velocity profiles are obtained only in the special case of a rotating two-dimensional disk in an axially symmetrical approach flow. This case was treated as the exact solution of the complete Navier-Stokes equations by D. M. Hannah in reference 8.

(e) Flow Over Plane and Wedge With Parabolic External Streamlines

In the case of a flow about a wedge, the velocity distribution $\overline{U} = A\xi^m$ appears in the direction perpendicular to the leading edge (ξ,η rectangular Cartesian coordinates). If a velocity component $\overline{V} = B\xi^n$ for $\xi > 0$ is added parallel to the front edge, the external streamlines will have a parabolic shape. According to the solution given in equations (5.8), the velocity profiles in the case of such a flow are "similar." The pertaining system of ordinary differential equations breaks up in this case; with introduction of a new scale factor $f^{*2} = \frac{1}{A} f^2$ it now reads

$$\left.\begin{aligned} F''' + \frac{m+1}{2} FF'' - m\left(F'^2 - 1\right) &= 0 \\ G''' + \frac{m+1}{2} FG'' - n(F'G' - 1) &= 0 \end{aligned}\right\} \qquad (6.6)$$

with boundary conditions: $F(0) = F'(0) = 0 \qquad F'(\infty) = 1$

$$G'(0) = 0 \qquad G'(\infty) = 1$$

The special case $m = 0$, $n = 1$ (parabolic flow over a plane plate) was treated by Loos in reference 15. For F' there is obtained of course in this case the ordinary Blasius plate profile. For the second component G', in contrast, an excess velocity results.

If the velocity vector is resolved, not in the direction of the rectangular Cartesian coordinates but into a component tangential to the direction of the external streamlines and into a component perpendicular to them, there are obtained for the primary flow component in the direction of the external streamlines for various values of the angle τ - formed by a streamline with the positive ξ direction - the velocity profiles shown in figure 10 which are, of course, no longer

"similar." If in the case of the flow now considered, for instance, streamline coordinates had been used, no "similar" velocity profiles would have been encountered. This illustrates the fact that the velocity profiles may very well be "similar," in the described sense, in the case of a prescribed flow with respect to a particular coordinate system whereas this need no longer be true with respect to any other coordinate system.

(f) Flows With Spiral-Shaped External Flows

If, in the solution indicated in equations (5.18) for $n = 1$, the grid of parameter lines are interpreted as two-dimensional polar coordinates, and $\overline{U}$ and $\overline{V}$ as two velocity components present in the external flow, there are obtained - because of $\overline{V} = k\overline{U}$ - external streamlines which form constant angles with the parameter lines. The external streamlines are, accordingly, logarithmic spirals. If especially $m = -1$ (vortex source) and $k = 1$, the pertaining differential equations read

$$\left.\begin{aligned} F''' + FF'' + F'^2 + G'^2 - 2 = 0 \\ G''' + FG'' = 0 \end{aligned}\right\} \tag{6.7}$$

with boundary conditions: $F(0) = F'(0) = 0 \qquad F'(\infty) = 1$

$$G'(0) = 0 \qquad G'(\infty) = 1$$

according to a remark of Prandtl in reference 20, numerical calculations he had performed did not lead to a unique solution. It is recognized that actually no uniquely determined solution for the boundary-value problem (6.7) is to be expected, if the transformation

$$\overline{F}' = F' - 1 \qquad \overline{G}' = G' - 1$$

is carried out and the equations for $\overline{F}$ and $\overline{G}$ are then linearized for large values of ζ. Thus, the equations

$$\overline{G}''' + \zeta\overline{G}'' = 0$$

$$\overline{F}''' + \zeta\overline{F}'' + 2\overline{F}' = -2\overline{G}' \quad \text{with} \quad \overline{F}'(\infty) = \overline{G}'(\infty) = 0$$

are obtained. The homogeneous part of the second equation is of the form of the differential equation

$$y'' + axy' + by = 0 \tag{6.8}$$

which, according to reference 26, p. 475, may be solved

$$y = \left(\frac{a}{2}\right)^{1/4} \cdot e^{-\frac{a}{2}x^2} \cdot \left\{\tilde{A}\left(\frac{a}{2}x^2\right)^{1/2} {}_1F_1\left(1 - \frac{b}{2a};\frac{3}{2};\frac{a}{2}x^2\right) + \tilde{B} \cdot {}_1F_1\left(\frac{1}{2} - \frac{b}{2a};\frac{1}{2};\frac{a}{2}x^2\right)\right\} = e^{-\frac{a}{2}x^2} \cdot \left\{Ax\ {}_1F_1\left(1 - \frac{b}{2a};\frac{3}{2};\frac{a}{2}x^2\right) + B \cdot {}_1F_1\left(\frac{1}{2} - \frac{b}{2a};\frac{1}{2};\frac{a}{2}x^2\right)\right\}$$

where ${}_1F_1$ signifies the confluent hypergeometrical function. Since ${}_1F_1$ has, according to reference 28, p. 271, the following asymptotic expansion:

$$y \approx e^{-\frac{a}{2}x^2}\left\{A\frac{\left(\frac{1}{2}\right)!\left(-\frac{a}{2}\right)^{\frac{b}{2a}-1}}{\left(-\frac{1}{2}+\frac{b}{2a}\right)!}x^{\frac{b}{a}-1}\left(1+\sum_1^\infty\frac{c_\mu^{(1)}}{x^{2\mu}}\right)+B\frac{-\left(\frac{1}{2}\right)!\left(-\frac{a}{2}\right)^{-\frac{1}{2}+\frac{1}{2a}}}{\left(\frac{b}{2a}-1\right)!}x^{1-\frac{b}{a}}\cdot\left(1+\sum_1^\infty\frac{c_\mu^{(3)}}{x^{2\mu}}\right)\right\}+$$

$$x^{-\frac{b}{a}}\cdot\left\{A\frac{(1/2)!}{\left(-\frac{b}{2a}\right)!}\left(\frac{a}{2}\right)^{-\frac{1}{2}-\frac{b}{2a}}\left(1+\sum_1^\infty\frac{c_\mu^{(2)}}{x^{2\mu}}\right)+B\frac{-(1/2)!}{\left(-\frac{1}{2}-\frac{b}{2a}\right)!}\left(\frac{a}{2}\right)^{-\frac{b}{2a}}\cdot\left(1+\sum_1^\infty\frac{c_\mu^{(4)}}{x^{2\mu}}\right)\right\}$$

the following asymptotic representation is obtained for y:

$${}_1F_1(\alpha;\beta;t) \approx \frac{(\beta-1)!}{(\beta-\alpha-1)!}(-t)^{-\alpha}\left\{1+\sum_{\mu=1}^\infty(-1)^\mu\frac{\alpha(\alpha+1)\ldots(\alpha+\mu-1)(\alpha-\beta+1)\ldots(\alpha-\beta+\mu)}{\mu!t^\mu}\right\}+$$

$$\frac{(\beta-1)!}{(\alpha-1)!}e^t t^{\alpha-\beta}\left\{1+\sum_{\mu=1}^\infty\frac{(1-\alpha)(2-\alpha)\ldots(\mu-\alpha)(\beta-\alpha)(\beta-\alpha+1)\ldots(\beta-\alpha+\mu-1)}{\mu!t^\mu}\right\}$$

Because of $c_\mu^{(1)} = c_\mu^{(3)}$, and $c_\mu^{(2)} = c_\mu^{(4)}$ (which can be easily checked), there is, furthermore,

$$y \approx e^{-\frac{a}{2}x^2} \cdot \left(1 + \sum_1^\infty \frac{c_\mu^{(1)}}{x^{2\mu}}\right) \cdot \left\{A \frac{\left(\frac{1}{2}\right)!\left(-\frac{a}{2}\right)^{\frac{b}{2a}-1}}{\left(-\frac{1}{2}+\frac{b}{2a}\right)!} x^{\frac{b}{a}-1} + B \frac{\left(-\frac{1}{2}\right)!\left(-\frac{a}{2}\right)^{-\frac{1}{2}+\frac{b}{2a}}}{\left(\frac{b}{2a}-1\right)!} x^{1-\frac{b}{a}}\right\} +$$

$$x^{-\frac{b}{a}} \cdot \left(1 + \sum_1^\infty \frac{c_\mu^{(2)}}{x^{2\mu}}\right) \cdot \left\{A \frac{\left(\frac{1}{2}\right)!\left(\frac{a}{2}\right)^{-\frac{1}{2}-\frac{b}{2a}}}{\left(-\frac{b}{2a}\right)!} + B \frac{\left(-\frac{1}{2}\right)!\left(\frac{a}{2}\right)^{-\frac{b}{2a}}}{\left(-\frac{1}{2}-\frac{b}{2a}\right)!}\right\}$$

Accordingly, for $a > 0$, $b > 0$ all solutions of the differential equation (6.8) tend toward zero, whereas for $a > 0$, $b < 0$ and for $a < 0$, $b = a$, a family of solutions tends toward zero. For $a < 0$, $b \neq a$, there do not exist any solutions which disappear at infinity.

Thus all solutions of the differential equation

$$\bar{F}''' + \zeta\bar{F}'' + 2\bar{F}' = C$$

as $\zeta \to \infty$ tend toward zero so that in the general solution of the equation for $\bar{F}'$ two additional free constants appear and therefore no uniquely determined solution of the boundary-value problem is to be expected.

If $m = 1$, a rotationally symmetrical stagnation-point flow is dealt with here, where the fluid, in addition, simultaneously rotates as a rigid body. The pertaining differential equations read

$$\left.\begin{aligned} F''' + 2FF'' - F'^2 + G'^2 &= 0 \\ G''' + 2FG'' - 2F'G' + 2 &= 0 \end{aligned}\right\} \qquad (6.9)$$

with boundary conditions: $F(0) = F'(0) = 0 \qquad F'(\infty) = 1$

$$G'(0) = 0 \qquad G'(\infty) = 1$$

For $\bar{F}' = F' - 1$, neglecting the quadratic terms, for large ζ the differential equation

$$\bar{F}''' + 2\zeta\bar{F}'' - 2\bar{F}' = 2\bar{G}'$$

is obtained. According to the above remarks, now only one fundamental solution of the homogeneous equation tends toward zero so that only one free constant appears in the asymptotic expansion of $\bar{F}'$ and a uniquely determined solution of the boundary-value problem (6.9) can be expected.

(g) Flows Over a Plane With the Velocity Distributions Indicated in Equations (5.4), (5.9), and (5.10)

The solution named in equations (5.4)

$$\bar{U} = A\xi^m\eta^n \qquad \bar{V} = B\xi^{m-1}\eta^{n+1} \qquad g_1 = g_2 = 1$$

can be interpreted as a flow over a plane where the equation of the streamlines reads

$$\eta = \alpha\xi^G \qquad C = \frac{B}{A}$$

Limiting ourselves to $A > 0$ (which does not mean a loss of generality since this case can always be attained by a rotation of the coordinate system), the pertaining differential equations may be written as follows:

$$\left.\begin{aligned} &F''' + \frac{m+1}{2}FF'' + C\cdot\frac{n+2}{2}GF'' + Cn(F'G' - 1) - m\left(F'^2 - 1\right) = 0 \\ &G''' + \frac{m+1}{2}FG'' + C\cdot\frac{n+2}{2}GG'' + (m-1)(F'G' - 1) - \\ &C(n+1)\left(G'^2 - 1\right) = 0 \end{aligned}\right\} \qquad (6.10)$$

with boundary conditions: $F(0) = F'(0) = 0 \qquad F'(\infty) = 1$

$$G(0) = G'(0) = 0 \qquad G'(\infty) = 1$$

The special case $m = 1$, $n = 0$ leads to the velocity distribution $\overline{U} = A\xi$, $\overline{V} = B\eta$ which - as L. Howarth has shown in reference 9 - appears in the neighborhood of a stagnation point on an arbitrary curvilinear surface. The above equations (6.10) were solved in reference 9 for this case, for various values of C.

If $\overline{U}$ and $\overline{V}$ show the velocity distributions given in equations (5.9) and (5.10)

$$\overline{U} = A \cdot e^{\alpha\xi} \qquad \overline{V} = B \cdot e^{\beta\xi}$$

and

$$\overline{U} = A \cdot e^{\alpha\xi} \qquad \overline{V} = (B \cdot \eta + D)e^{\alpha\xi}$$

the external streamlines are exponential curves, in the first case of the form

$$\eta = C \cdot e^{\delta\xi} + \text{Constant} \qquad C = \frac{B}{A}\frac{1}{\beta - \alpha} \qquad \delta = \beta - \alpha$$

in the second case of the form

$$\eta = \text{Constant} \cdot e^{\frac{B}{A}\cdot\xi}$$

7. "SIMILAR" BOUNDARY LAYERS ON BODIES OF REVOLUTION WHICH ROTATE ABOUT THEIR AXIS IN A FLUID OTHERWISE AT REST

(a) Shape of Bodies of Revolution Pertaining to "Similar" Solutions

In the two preceding sections it has been shown, among other things, that "similar" boundary layers occur on bodies of revolution where the distance from the axis of rotation equals a power of the arc length η of the meridian curve. The form of the meridian curves for which this is true may be determined as follows:

If x_1, x_2 are rectangular Cartesian coordinates made dimensionless with L, there applies for the arc length $\eta = y/L$ of the meridian curve

$$\eta = \eta(x_1) = \int_{x_{10}}^{x_1} \sqrt{1 + h'^2(t)}\, dt + \eta_0$$

if $x_2 = h(x_1)$ is the equation of the meridian curve in the x_1x_2 coordinate system. Hence follows:

$$\left(\frac{d\eta}{dx_1}\right)^2 = 1 + \left(\frac{dh}{dx_1}\right)^2$$

Because of

$$\bar{r}\left(\eta(x_1)\right) = h(x_1) \tag{7.1}$$

there is obtained

$$\frac{dh}{dx_1} = \frac{d\bar{r}}{d\eta} \cdot \frac{d\eta}{dx_1}$$

Thus,

$$\left(\frac{d\eta}{dx_1}\right)^2 = 1 + \left(\frac{d\bar{r}}{d\eta}\right)^2 \cdot \left(\frac{d\eta}{dx_1}\right)^2$$

whence the following relationship between x_1 and $\bar{r}$ is obtained:

$$x_1 = x_1(\eta) = \int_{\eta_0}^{\eta} \sqrt{1 - \left(\frac{d\bar{r}}{dt}\right)^2}\, dt + x_{10} \tag{7.2}$$

If the inverse function $\eta = \eta(x_1)$ is formed and inserted into equation (7.1), the desired equation of the meridian curve is obtained immediately.

Especially for $\bar{r} = A\eta^m$, equation (7.2) reads

$$x_1 = \int_{\eta_0}^{\eta} \sqrt{1 - A^2m^2t^{2(m-1)}}\, dt + x_{10} \tag{7.3}$$

The value of this integral may be given in closed form only for individual special values of m; the rest are dependent on numerical evaluation.

For different values of m the forms of meridian curves indicated in figure 14 are obtained. It is noted that the latter are real for $m < 1$ only from a certain point P_0 onward (or, for $m > 1$, only up to a certain point P_0). As a consequence, the pertaining bodies of revolution have an opening in front or do not extend to infinity at the rear. All considerations made until now presuppose surfaces which have no edges whatever. The solution of the system of equations (6.5) will therefore not satisfy the boundary conditions added due to the edges mentioned.

In the case $0 < m < 1$, however, and for sufficiently small values of A, the radius $\bar{r}_0$ of the opening present at the front of the pertaining body of revolution is so small that the opening will have no noticeable influence on the flow conditions so that, except for the immediate vicinity of the leading edge, the solution of (6.5) may be regarded as valid. Likewise, the solution of (6.5) can unhesitatingly be used in the case $m > 1$ if only the dimensions of the body considered are large compared to the thickness of the boundary layer. The immediate neighborhood of the point at the front of these bodies of revolution is, of course, to be excluded from the consideration.

For $m = 1$, cones as bodies of revolution are obtained. The parameter A and the half vertex angle ϑ of the cone of revolution are in the relation $A = \sin \vartheta$. Since A does not enter into the differential equations for F and G describing the boundary-layer flow, the boundary layers pertaining to cones of revolution with different vertex angles differ only by their thickness. Since for $A = 1$ in particular the two-dimensional disk is obtained as body of revolution, the results from the calculation of the flow near a rotating disk to circular cones may be transferred immediately and the following result may be derived:

In the case of a cone of revolution rotating in a fluid otherwise at rest, the component of the centrifugal force parallel to the wall causes a secondary flow in the direction of the generating-surface lines of the cone rearward, whereas, at the outer edge of the boundary layer, for reasons of continuity, fluid flows in simultaneously.

It should be noted that cones of revolution with very small vertex angles are to be excluded here since for such cones the boundary-layer assumptions no longer hold true.

(b) Elementary Proof for the Existence of a Solution of

the Pertaining Boundary-Value Problem

In section 4 it was shown that the system of equations (3.5) in combination with the boundary conditions (3.5a) has a solution only in the case where the coefficients appearing in the differential equations are constant. Every single case must therefore be separately examined to see whether a solution actually exists.

If, for instance, $d = -1$ is chosen, and all other coefficients equal zero, the differential equation $F''' = F'^2 - 1$ is obtained, which has no solution satisfying the boundary conditions $F'(0) = 0$, $F'(\infty) = 1$. (See ref. 19.)

In the case considered now, of the flow in the neighborhood of rotating bodies of revolution which is described by the equations

$$\left.\begin{aligned} F''' + cGF'' - 2F'G' &= 0 \\ G''' + cGG'' - G'^2 + F'^2 &= 0 \end{aligned}\right\} \quad \text{with} \quad c = \text{Constant} > 0 \tag{6.5}$$

with the boundary conditions: $F'(0) = 1$ $\qquad F'(\infty) = 0$

$$G(0) = G'(0) = 0 \qquad G'(\infty) = 0$$

the existence of a solution can be proved.

The following proof is carried out with generalization of the method applied to the boundary-value problem

$$F''' + FF'' + d\left(F'^2 - 1\right) = 0 \qquad F(0) = C \qquad F'(0) = 0 \qquad F'(\infty) = 1$$

in a paper of R. Iglisch (ref. 24). This method is essentially based on investigating the behavior of all solutions in the vicinity of the zero point which satisfy the boundary conditions at the zero point, and on showing that the limiting curve of all solutions with a particular property is, on the basis of elementary considerations of shape, the solution of the boundary-value problem.

For performing the proof, the transformation $\bar{F}' = 1 - F'$ is carried out so that the present boundary-value problem assumes the following form:

$$\left.\begin{aligned} &\bar{F}''' + c\bar{F}\bar{F}'' + 2G'(1 - \bar{F}') = 0 \\ &G''' + cGG'' - G'^2 + (1 - \bar{F}')^2 = 0 \end{aligned}\right\} \qquad (7.5)$$

with boundary conditions: $\bar{F}'(0) = 0$ $\qquad$ $\bar{F}'(\infty) = 1$

$$G(0) = G'(0) = 0 \qquad G'(\infty) = 0$$

Since only the function $\bar{F}'$ is used in what follows, the bars above $\bar{F}'$ and its derivatives will again be omitted.

Let

$$F''(0) = A \qquad G''(0) = B$$

The proof then results on the basis of the following theorems:

Theorem (1).- For an arbitrary fixed $B > 0$ and a sufficiently large A, the pertaining solution F' intersects the straight line $F' = 1$; there is always $F''' < 0$ as long as $F' \leqq 1$ so that F' can show a turning point only for $F' > 1$.

Theorem (2).- For an arbitrary fixed $B > 0$ and a sufficiently small $A > 0$, the pertaining solution F' has at least one turning point, as long as $F' < 1$ or does not intersect the straight line $F' = 1$ at all. For fixed $B > 0$ let A^* be the upper limit of all A-values whose pertaining solutions F' intersect the straight line $F' = 1$ and have a turning point, as long as $F' < 1$, or do not intersect the straight line $F' = 1$ and have a maximum <1. Such a limit exists, because of theorem (1), for every fixed $B > 0$. For the solution $F^{*\prime}$ pertaining to A^* there then applies:

Theorem (3).- Either $F^{*\prime}$ intersects the straight line $F' = 1$ and has a turning point at the point of intersection with this straight line; or $F^{*\prime}$ intersects the straight line $F' = 1$ curved to the right ($F^{*'''}$ has then at least one zero as long as $F^{*\prime} \leqq 1$); or $F^{*\prime}$ does not intersect the straight line $F' = 1$ at all.

Theorem (4).- The solution $G^{*\prime}$ pertaining to the pair of initial values (A^*,B) has negative values for sufficiently small $B > 0$.

Theorem (5).- For a sufficiently large B, $G^{*\prime}$ always remains positive.

Theorem (6).- The pair of solutions F_L', G_L' pertaining to the initial values (A_L, B^*) is the solution of the boundary-value problem. B^* is here defined as the upper limit of all B-values for which $G^{*\prime}$ becomes negative. Such a limit exists because of theorem (5). A_L coincides with A^* if the function $A^* = A^*(B)$, for $B = B^*$, is continuous. If it is not, A_L is a value between A^* and $\bar{A}^*$ to be defined more exactly, with $\bar{A}^*$ being the lower limit of all A-values which are $\geqq A^*$ and for which the pertaining solution F' has the property specified in theorem (1).

That the solutions employed for the following proof actually exist, as far as they are used, is immediately clear on the basis of the existence theorem for systems of ordinary differential equations of the first order[6] if care is taken that with F', G, and G', for all finite ζ, F'' and G'', too, remain limited. This latter result is obtained if the two differential equations are solved with respect to F''' or G''', and then integrated with respect to ζ. Thus,

$$F''(\zeta) - A = -c \int_0^\zeta GF''dt - 2G + 2 \int_0^\zeta F'G'dt$$

$$= -G(2 + cF') + (2 + c) \cdot \int_0^\zeta F'G'dt$$

and

$$G''(\zeta) - B = -c \int_0^\zeta GG''dt + \int_0^\zeta G'^2dt \cdot \int_0^\zeta (1 - F')^2dt$$

$$= -cGF' + (1 + c) \int_0^\zeta G'^2dt - \int_0^\zeta (1 - F')^2dt$$

are obtained.

Now that this preliminary remark has been made, proof of the separate theorems will be taken up.

---

[6]See reference 25, p. 124.

Theorem (1).-

Auxiliary theorem 1, (1): A number $\zeta_{11}(B) > 0$ exists in such a manner that the solution of the differential equation $G''' + cGG'' - G'^2 = 0$ with $G(0) = G'(0) = 0$, $G''(0) = B$ remains limited in J_{11}.[7] This follows immediately from the existence theorem for systems of ordinary differential equations.

Auxiliary theorem 1, (2): For a fixed B, there exists a number $\zeta_{12}(B) > 0$ valid for all $A > 0$ in such a manner that F' has no maximum in J_{12}, as long as $F' \leqq 1$.

For proof, let F' have a first maximum for $\zeta = \zeta_{13}$; that is, let $F''(\zeta_{13}) = 0$, and let $0 \leqq F' \leqq 1$ in J_{13}. From $F''' = -cGF'' - 2G' + 2F'G'$ there follows then by integration from 0 to ζ_{13}, with consideration of the boundary conditions for $\zeta = 0$

$$-A = -cG(\zeta_{13}) \cdot F'(\zeta_{13}) - 2G(\zeta_{13}) + (2 + c) \int_0^{\zeta_{13}} F'G' \cdot dt \qquad (7.6)$$

If no interval of the kind named exists, the point ζ_{13} must shift toward zero if A grows without limit. In particular, ζ_{13} somewhere becomes $< \zeta_{11}$. However, then the solution of the differential equation

$$G''' + cGG'' - G'^2 = 0$$

with

$$G(0) = G'(0) = 0 \qquad G''(0) = B$$

remains uniformly limited for all A in the entire interval J_{13}. The same is true on the basis of the theorem regarding the dependence of the integrals on the right side of the differential equations;[8] also for the solution of the differential equation

---

[7] In the course of the following proof, $J_{\mu\nu}$ will always signify the interval $0 \leqq \zeta \leqq \zeta_{\mu\nu}$.

[8] See reference 25, p. 153.

$$G''' + cGG'' - G'^2 = -(1 - F')^2$$

for the same initial conditions, since in J_{1z} everywhere $0 \leqq (1 - F') \leqq 1$. If A is allowed to increase without limit, the above relationship (7.6) therefore yields directly a contradiction.

Auxiliary theorem 1, (3): At the point of a maximum of G' there must exist $|G'| \leqq |1 - F'|$.

This results directly from the second differential equation of (7.5) which otherwise would show a contradiction in sign.

Auxiliary theorem 1, (4): For an arbitrary fixed $B > 0$ and a sufficiently large A, there is obtained $F' > 1$ in an interval

$$0 \leqq \zeta \leqq \zeta_{10} < \bar{\zeta}_1$$

with

$$\bar{\zeta}_1 = \bar{\zeta}_1(B) = \mathrm{Min}\left(1, \frac{2}{ck_1}, \zeta_{11}, \zeta_{12}\right)$$

Let the proof be carried out indirectly. If it is assumed that $F' \leqq 1$ in J_{10}, there applies because of $\zeta_{10} \leqq \zeta_{12}$ in J_{10} the inequality $|1 - F'| \leqq 1$ so that the solution of the differential equation

$$G''' + cGG'' - G'^2 = -(1 - F')^2$$

with

$$G(0) = G'(0) = 0 \qquad G''(0) = B$$

certainly remains limited. Let it be assumed, for instance, that $G' < k_1(B)$ in J_{10}. Because $\zeta_{10} < 1$, there is also $G < k_1$. As long as the inequality $0 \leqq F'' \leqq A$ is satisfied, there applies therefore

$$-cGF'' > -ck_1A \quad \text{and} \quad -2G'(1 - F') > -2k_1$$

thus

$$F''' = -cGF'' - 2G'(1 - F') > -ck_1A - 2k_1 = -k_1(2 + cA) = -\tilde{k}_1$$

with $\tilde{k}_1 = \tilde{k}_1(A,B) > 0$. From the above inequality there follows by integration

$$F'' \geqq A - \tilde{k}_1\zeta \tag{7.7}$$

Because of $\tilde{k}_1 > 0$, this inequality exists first just for $F'' > A$, accordingly, always as long as $F'' \geqq 0$, that is, at least in the interval J_{15} with

$$\zeta = \text{Min}\left(\zeta_{10}, \zeta_{14}\right)$$

where ζ_{14} is defined by[9]

$$A - \tilde{k}_1\zeta_{14} = 0$$

By integration of the inequality (7.7) valid in J_{15}

$$F' \geqq A\zeta - \frac{\tilde{k}_1}{2}\zeta^2$$

is now obtained. Hence follows

(a) For $\zeta_{14} \leqq \zeta_{10}$

$$F'\left(\zeta_{15}\right) = F'\left(\zeta_{14}\right) \geqq A\zeta_{14} - \frac{\tilde{k}_1}{2}\zeta_{14}^2 = \frac{A^2}{\tilde{k}_1} - \frac{\tilde{k}_1}{2} \cdot \frac{A^2}{\tilde{k}_1^2} = \frac{A^2}{2\tilde{k}_1} = \frac{1}{2}\frac{A^2}{k_1(2 + cA)}$$

---

[9] If F'' were $\geqq 0$ only up to a value $\tilde{\zeta}_{14} < \zeta_{14}$, thus $F''\left(\tilde{\zeta}_{14}\right) = 0$, the inequality (7.7) would be valid only in the interval $0 \leqq \zeta \leqq \tilde{\zeta}_{14}$. For $\zeta = \tilde{\zeta}_{14}$ there would then result:

$$0 = F''\left(\tilde{\zeta}_{14}\right) \geqq A - \tilde{k}_1\tilde{\zeta}_{14} > A - \tilde{k}_1\zeta_{14} = 0$$

that is, a contradiction would arise.

that is, for a sufficiently large A, $F'(\zeta_{15}) > 1$ is obtained.

(b) For $\zeta_{14} > \zeta_{10}$

$$F'(\zeta_{15}) = F'(\zeta_{10}) \geq A\zeta_{10} - \frac{\tilde{k}_1}{2}\zeta_{10}^2 = \left(1 - \frac{ck_1}{2}\zeta_{10}\right)A\zeta_{10} - k_1\zeta_{10}^2$$

Thus $F'(\zeta_{15}) > 1$ is also obtained, if only $A > \frac{1 + k_1\zeta_{10}^2}{a\zeta_{10}}$.

The assumption $F' \leq 1$ leads, therefore, to a contradiction for sufficiently large A; thus the auxiliary theorem 1, (4) is proved. The proof of theorem (1) itself now results as follows: Let it be assumed that

$$B - \frac{K_1}{2}\zeta_{16} = 0$$

thus $\zeta_{16} = \frac{2B}{K_1}$ with

$$K_1 = K_1(B) = 1 + cB > 0$$

Furthermore, assume that

$$\zeta_{17} = \text{Min}(\zeta_{16}, \bar{\zeta}_1)$$

According to the auxiliary theorem 1, (4) just proved, $F' > 1$ is obtained in the interval J_{17} for a sufficiently large A. If $F'(\zeta)$ is a solution for which the latter fact holds true, and if ζ_{18} is the abscissa of the intersection point with the straight line $F' = 1$, G' and, therefore, also G will always be positive in the interval $0 < \zeta \leq \zeta_{18}$. This can be understood as follows: From the second equation of (7.5) there follows for the interval J_{18} - where of course $0 \leq F' \leq 1$ is valid - the estimate

$$G''' = -cGG'' + G'^2 - (1 - F')^2 \geq -cGG'' + G'^2 - 1 \geq -(1 + cGG'')$$

If somewhere in the interval considered $G' > 1$ is obtained, no maximum of G' can appear anymore, according to auxiliary theorem 1, (3); that is, G' is then certainly positive in the entire interval. Thus there

remains only the case to be considered that G' in J_{18} is always $\leqq 1$. G is then likewise $\leqq 1$. As long as $G'' \leqq B$ and $G' \geqq 0$,

$$G''' \geqq -1 - cGG'' > -(1 + cB) = -K_1$$

from which by integration

$$G'' \geqq B - K_1\zeta$$

is obtained. Because $K_1 > 0$, this inequality is valid all the more for $G'' > B$, thus always, as long as $G' \geqq 0$. By integrating once more

$$G' \geqq B\zeta - \frac{K_1}{2}\zeta^2 = \zeta\left(B - \frac{K_1}{2}\zeta\right) > \zeta\left(B - \frac{K_1}{2}\zeta_{15}\right) = 0$$

is obtained; that is, in $0 < \zeta \leqq \zeta_{18}$ there is everywhere $G' > 0$ and thus also $G > 0$.

Since

$$F''' = -cGF'' - 2G'(1 - F')$$

it follows that everywhere in $0 < \zeta \leqq \zeta_{18}$ there exists $F''' < 0$ as long as $F'' \geqq 0$. However, F'' cannot become negative in the interval considered, since in that case because

$$F'\left(\zeta_{18}\right) = 1$$

a minimum of F' would have to follow, which obviously would lead to a contradiction in sign in the first equation of (7.5). Thus, in the entire interval $0 < \zeta \leqq \zeta_{18}$, $F''' < 0$. Thereby theorem (1) is proved completely.

Theorem (2).- The proof of theorem (2) is based on the statement: The solution F' pertaining to the initial values $A = 0$, $B > 0$ (fixed) assumes, in a certain vicinity of the zero point, negative values.

Because

$$F^{IV}(0) = -2B$$

there exists in a certain vicinity J_{21} of the zero point $F^{IV}(\zeta) \leqq -B$; hence follows, with consideration of the boundary conditions at the zero point: $F' \leqq -B\zeta^3$. In particular, therefore, $F'(\zeta_{21}) \leqq -B\zeta_{21}^3$.

The solution F' pertaining to the initial values $A > 0$, $B > 0$, with sufficiently small A, must also be negative at the point ζ_{21} because of the continuous dependence of the solutions on the initial conditions.[10] Because

$$F''(0) = A > 0$$

this is possible only when F' has a maximum <1 in the interval J_{21}. If the solutions of this type intersect the straight line $F' = 1$ at all at any time, they must therefore, for geometrical reasons, show at least one turning point as long as $F' < 1$, whereby the statement of theorem (2) is proved.

Theorem (3).- The differential equations considered may be reduced to a system of five differential equations of the first order in the known manner. Application of the theorem on the dependence of the solutions of a system of ordinary differential equations of the first order on the initial conditions yields the statement that F', F'', G, G', and G'' depend continuously on the initial conditions; on the basis of the theorem regarding the continuity of an averageable function, F''' and G''' they depend likewise continuously on the initial conditions. On this fact is founded the following:

Auxiliary theorem 3, (1): The ordinate of a turning point of F' varies continuously with the initial conditions. For proof of this theorem let $\tilde{F}'$ be the solution pertaining to the initial values $\tilde{A}$, $\tilde{B}$; furthermore, let $\tilde{F}'$ have a turning point at the point ζ_{30}:

$$\tilde{F}'''(\zeta_{30}) = 0$$

Let

$$\tilde{F}'(\zeta_{30}) = \alpha$$

---

[10]See reference 25, p. 152.

If the given theorem were not correct, an ϵ_0 would exist in such a manner that $\tilde{F}'' \neq 0$ and $\tilde{F}''' \neq 0$ as long as F' lies in the strip

$$\alpha - \epsilon_0 \leqq \tilde{F}' \leqq \alpha + \epsilon_0$$

and that for all solutions, pertaining to the initial values $(\tilde{A} + \delta, \tilde{B} + \delta)$, $F''' \neq 0$, as long as they lie in the strip named, however small δ may have been chosen. If $\tilde{F}'$ attains at the point ζ_{31} the functional value $\beta = \alpha + \frac{\epsilon_0}{2}$, and if $\tilde{F}'''(\zeta_{31}) = \gamma \neq 0$, there exists, because of the continuous dependence on the initial conditions, a δ_1 in such a manner that

$$\left|\tilde{F}'''(\zeta_{31}) - F'''(\zeta_{31})\right| < \gamma$$

and a δ_2 in such a manner that

$$\left|F'(\zeta_{31}) - \tilde{F}'(\zeta_{31})\right| < \frac{\epsilon_0}{2}$$

For

$$\delta = \text{Min}(\delta_1, \delta_2)$$

the pertaining solution F' for $\zeta = \zeta_{31}$ then shows the same sign of F''' as $\tilde{F}'''$, so that F' certainly has a turning point as long as $F' < \alpha + \epsilon_0$. By means of a corresponding consideration of the point ζ_{32} for which

$$\tilde{F}'(\zeta_{32}) = \alpha - \frac{\epsilon_0}{2}$$

it is then recognized that the turning point must lie above the straight line $F' = \alpha - \epsilon_0$; thus the auxiliary theorem indicated above is proved.

On the basis of this auxiliary theorem the proof of theorem (3) results easily.

For the variation of $F^{*\prime}$ the following possibilities are conceivable:

(a) $F^{*\prime}$ intersects the straight line $F' = 1$ and has

(α) A turning point as long as $F^{*\prime} < 1$

(β) A turning point at the intersection point with the straight line $F' = 1$

(γ) Only one turning point if $F^{*\prime} > 1$

(b) $F^{*\prime}$ is tangent to the straight line $F' = 1$.

(c) $F^{*\prime}$ does not intersect the straight line $F' = 1$ at all.

Since the solutions pertaining to all initial values $A > A^*$ have a turning point only for $F' > 1$, the case $a(\alpha)$ cannot occur, on the basis of the auxiliary theorem 3, (1). Likewise, the case $a(\gamma)$ is possible only when $F^{*\prime\prime\prime}$ disappears at least once as long as $F^{*\prime} < 1$. (The points on $F^{*\prime}$ where $F^{*\prime\prime\prime}$ disappears are limiting positions of two turning points coming together.) The case (b), in turn, cannot occur since the solution $F' \equiv 1$ pertaining to the initial conditions $F' = 1$, $F'' = 0$ is uniquely determined. Thus there remain only the possibilities $a(\alpha)$, $a(\gamma)$, and (c); this is precisely the statement of theorem (3). Moreover, the following facts are recognized: $F^{*\prime}$ cannot have a maximum if $F^{*\prime}$ does not intersect the straight line $F' = 1$, that is, $F^{*\prime}$ is always monotonically increasing in this case. (If $F^{*\prime}$ had a maximum, there would exist - because of the continuous dependence of the solutions on the initial conditions for sufficiently small δ - solutions which likewise have a maximum <1 also for the initial values $(A^* + \delta, B)$; however, on the basis of the definition of A^* this is not possible.)

In the case $a(\alpha)$, G must disappear at the point of intersection of $F^{*\prime}$ with the straight line $F' = 1$. This follows immediately from the first differential equation of (7.5). $G^{*\prime}$ must therefore assume negative values in the interval in which $F^{*\prime}$ lies between 0 and 1. The same is true for the case $a(\gamma)$ since, at the point where $F^{*\prime\prime\prime}$ disappears, G and $G^{*\prime}$ must have opposite signs.

Theorem (4).- In the proof of the auxiliary theorem 1, (4) the quantities $\bar{\zeta}_1$ and k_1 were still dependent on B. However, it is recognized immediately that two positive numbers $\bar{\bar{\zeta}}_1$ and $\bar{\bar{k}}_1$ may be found which are valid for all values of B where $0 \leqq B \leqq 1$ so that also an A_4 may be indicated in such a manner that, for all values of $A \geqq A_4$ and for all values of B where $0 \leqq B \leqq B_4 < 1$, F' becomes >1 in $0 \leqq \zeta \leqq \zeta_{10} < \bar{\bar{\zeta}}_1$.

Now let $A \geqq A_4$ be a specified constant. If B is then chosen sufficiently small, F' has at least one turning point below the straight line $F' = 1$. This can be explained as follows:

First, a number ζ_{40} valid for all values of B where $0 \leqq B \leqq B_4$ exists in such a manner that $(1 - F')^2 \geqq \frac{1}{2}$ in J_{40}. Furthermore there exists uniformly, for all values of B where $0 \leqq B \leqq B_4$, an interval J_{41} in which $-cGG'' + G'^2 \leqq \frac{1}{4}$. In

$$0 \leqq \zeta \leqq \zeta_{42} = \operatorname{Min}\left(\zeta_{40}, \zeta_{41}\right)$$

there then applies the estimate

$$G''' = -cGG'' + G'^2 - (1 - F')^2 \leqq -cGG'' + G'^2 - \frac{1}{2} \leqq \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

Hence follows by integration

$$G'' \leqq B - \frac{\zeta}{4}$$

$$G' \leqq B\zeta - \frac{\zeta^2}{8} = \zeta\left(B - \frac{\zeta}{8}\right)$$

$$G \leqq \frac{B}{2}\zeta^2 - \frac{\zeta^3}{24} = \frac{\zeta^2}{2}\left(B - \frac{\zeta}{12}\right)$$

For $B < \bar{B}_4 = \frac{\zeta_{42}}{12}$, G' and G thus become negative for $\zeta = \zeta_{42}$. Therefore $F'''\left(\zeta_{42}\right) > 0$. Because $F'''(0 + \epsilon) < 0$, F''' thus has a turning point in J_{42}. The solution F' pertaining to the pair of initial values $\left(A_4, \hat{B}_4\right)$, with $0 < \hat{B}_4 < \bar{B}_4$, intersects, therefore, the straight line $F' = 1$ and has a turning point with an ordinate value <1. According to the definition of A^*, $A^*\left(\hat{B}_4\right) > A_4$, that is, the solution $F^{*\prime}$ pertaining to the initial values $\left(A^*, \hat{B}_4\right)$ intersects the straight line $F' = 1$. According to the remarks made regarding theorem (3), $G^{*\prime}$ must then assume negative values whereby theorem (4) is proved.

It is recognized moreover from the above reasoning that the A*-values increase without limit for $B \to 0$ since, otherwise, a contradiction to theorem (1) would result.

Theorem (5).- For proving theorem (5), consider the solutions $F^{*\prime}$, G^* pertaining to the initial values (A^*,B) with

$$B > B_5 = 2 + c$$

Assume that

$$0 \leqq (1 - F^{*\prime}) \leqq 1 \quad \text{in} \quad 0 \leqq \zeta \leqq \zeta_{50}$$

and

$$\zeta_{51} = \text{Min}\left(\zeta_{50}, 1\right)$$

Then $G^{*\prime} > 1$ in J_{51}. It is simplest to perform the proof for this indirectly. If it is assumed that in the interval considered $G^{*\prime} \leqq 1$, it follows that there also $G^* \leqq 1$, because $\zeta_{51} \leqq 1$. Thus there applies, as long as $G^{*\prime\prime} > 0$

$$G^{*\prime\prime\prime} = -cG^*G^{*\prime\prime} + G^{*\prime 2} - (1 - F^{*\prime})^2 \geqq -cG^{*\prime\prime} - 1$$

Hence follows by integration

$$G^{*\prime\prime} \geqq B - cG^{*\prime} - \zeta \geqq B - c - 1 > B_5 - c - 1 = 1$$

that is, $G^{*\prime\prime} > 1$ everywhere in J_{51}. Hence follows $\zeta_{51} = 1$. If ζ_{51} were equal to ζ_{50}, $F^{*\prime}$ would intersect the straight line $F' = 1$ at the point ζ_{51} and $G^{*\prime}$ would therefore become negative in J_{51}; this, however, is not possible on the basis of the above estimate for $G^{*\prime\prime}$. Thus,

$$\zeta_{51} = 1$$

and therefore

$$G^{*\prime} > 1 \quad \text{for} \quad \zeta = \zeta_{51}$$

which is in contradiction to the assumption $G^{*\prime} \leqq 1$ in J_{51}.

Once $G^{*\prime}$ is >1, no maximum of $G^{*\prime}$ can appear anymore, according to the auxiliary theorem 1, (3), as long as $0 \leqq F^{*\prime} \leqq 1$ is valid. Thus $G^{*\prime}$ is monotonically increasing, and $G^{*\prime}$ and G^{*} are therefore always positive. Since, on the other hand, $F^{*\prime}$ can become >1 only when $G^{*\prime}$ has been negative somewhere before, there follows immediately that theorem (5) is correct.

Theorem (6).- For proving theorem (6) let the continuousness of the function $A^{*}(B)$ for $B = B^{*}$ be presupposed. For the pair of solutions F_L', G_L' pertaining to the initial values (A^{*},B^{*}) there then applies:

Auxiliary theorem 6, (1): G_L' always remains positive. Once G_L' became negative, solutions which somewhere assume negative values would exist also for the initial values $(A^{*},B^{*} + \delta)$ (on the basis of the theorem regarding the continuous dependence of the solutions on the initial conditions for sufficiently small δ) which is not possible on the basis of the definition of B^{*}.

Auxiliary theorem 6, (2): F_L', monotonically increasing, approaches an asymptote $F' = p_1$ with $0 \leqq p_1 \leqq 1$. This follows immediately from the additional remarks on theorem (3) since, with G_L', G_L also remains always positive.

Auxiliary theorem 6, (3): G_L' lies entirely in the strip $0 \leqq G_L' \leqq 1$. The left part of this inequality follows from the auxiliary theorem 6, (1), the right part with application of the auxiliary theorem 1, (3): If G_L' were at one point >1, all solutions $G^{*\prime}$ pertaining to $(A^{*},B^{*} - \delta)$ would become >1 also for sufficiently small δ, as long as $F^{*\prime} < 1$. (For $\delta \to 0$, the intersection points of $F^{*\prime}$ with the straight line $F' = 1$ - if such intersection points exist at all - shift to infinity, on the basis of the continuous dependence of the solutions on the initial conditions.) However, because of auxiliary theorem 1, (3) this is not possible.

Auxiliary theorem 6, (4): Once G_L' has a minimum, no maximum can follow anymore. At the point of a minimum where $G_L''' > 0$ and $G_L'' = 0$,

there must be $G_L' > (1 - F_L')$, on the basis of the second differential equation of (7.5), whereas at the point of a maximum the inequality $G_L' < (1 - F_L')$ must be satisfied; hence the statement of the auxiliary theorem follows immediately since $(1 - F_L')$ is monotonically decreasing. Moreover, it is recognized that G_L' can have no zero since $(1 - F_L')$ is positive for all finite values of ζ. Thus the two following possibilities for the asymptotic variation of F_L', G_L' result:

(a) $(1 - F_L')$, decreasing monotonically, approaches an asymptote $p_1 \geqq 0$. G_L', increasing monotonically, approaches an asymptote $q_1 \leqq 1$.

(b) $(1 - F_L')$, decreasing monotonically, approaches an asymptote $p_1 \geqq 0$. G_L', decreasing monotonically, approaches an asymptote $q_1 \geqq 0$.

Auxiliary theorem 6, (5): $p_1 = 0$ is valid. (F_L' cannot, increasing monotonically, approach a horizontal asymptote $F' = p_1$ with $0 \leqq p_1 < 1$). For proof:

(a) Let G_L remain limited for $\zeta \to \infty$. Then G_L' certainly tends toward zero. The same is true for G_L'', since G_L'' always has from a certain point onward the same sign. From the second differential equation of equations (7.5) then follows, because $(cG_LG_L'' - G_L'^2) \to 0$ that, from a certain point onward, G_L''' is always negative when $(1 - F_L')$ tends toward $p_1 > 0$ which is, however, not possible because $G_L'' \to 0$.

(b) Let G_L tend toward infinity for $\zeta \to \infty$. In this case it is expedient to perform, for the proof, the following transformation (which is justified because $G_L' > 0$):

$$\left.\begin{aligned} p(G_L) &= \left\{1 - F_L'\left(\zeta(G_L)\right)\right\} \\ q(G_L) &= G_L'\left(\zeta(G_L)\right) \end{aligned}\right\} \qquad (7.8)$$

Then there is obtained, where derivatives with respect to G_L are denoted by a dot:

$$-F_L'' = q\dot{p} \qquad -F_L''' = q^2\ddot{p} + q\dot{q}\dot{p}$$

$$G_L'' = q\dot{q} \qquad G_L''' = q^2\ddot{q} + q\dot{q}^2$$

Between ζ and G_L there exists the relationship

$$\zeta = \int_0^{G_L} \frac{dG_L}{q(G_L)}$$

Because $q \neq 0$, there is thus obtained for p and q the following system of differential equations:

$$\left.\begin{aligned} q\ddot{p} + (cG_L + \dot{q})\dot{p} - 2p = 0 \\ q^2\ddot{q} + q(cG_L + \dot{q})\dot{q} - q^2 + p^2 = 0 \end{aligned}\right\} \tag{7.9}$$

Since, according to assumption, G_L tends toward infinity for $\zeta \to \infty$, both p and q can no longer have a change in sign for sufficiently large G_L; for this reason, $\dot{p} \to 0$ and $\dot{q} \to 0$ must be valid for $G_L \to \infty$. For sufficiently large values of G_L, $(cG_L + \dot{q})$ thus certainly becomes positive; hence follows, because $\dot{p} < 0$, $p > 0$ that, from a certain point onward, $\ddot{p}$ is always positive. Because $\dot{p} \to 0$, there must also exist $\ddot{p} \to 0$ for $G_L \to \infty$ whence $p \to 0$ results on the basis of the first equation of equations (7.9).

Auxiliary theorem 6, (6): $q_1 = 0$ is valid. (G_L' cannot, monotonically increasing or monotonically decreasing, approach a horizontal asymptote $G' = q_1$ with $0 < q_1 \leqq 1$). For proof:

(a) First there will be shown a contradiction for the case where G_L', increasing monotonically, approaches the asymptote $G' = q_1$ with $0 < q_1 \leqq 1$ (case (a) of auxiliary theorem 6, (4)).

With the initial values $(A^*, B^* - \delta)$, even for arbitrarily small δ, there still exist solutions $G^{*\prime}$ which assume negative values; because of the continuous dependence of the solutions on the initial conditions this is possible, for the assumed variation of G', only by the shifting to infinity of the maxima of these solutions for $\delta \to 0$. Because $(1 - F_L') \to 0$ and $G_L' \to q_1$ for $\zeta \to \infty$, there exists a ζ_6 in such a

manner that $(1 - F_L') < \frac{q_1}{4}$ and $(G_L' - q_1) < \frac{q_1}{4}$ for all $\zeta \geq \zeta_6$. For a sufficiently small δ, the maximum of $G^{*\prime}$ then lies to the right of ζ_0, and there applies moreover

$$\left|G^{*\prime}(\zeta_0) - G_L'(\zeta_0)\right| < \frac{q_1}{4} \qquad \left|F^{*\prime}(\zeta_0) - F_L'(\zeta_0)\right| < \frac{q_1}{4}$$

Thus,

$$\left|G^{*\prime} - q_1\right| < \frac{q_1}{2} \qquad \left|1 - F^{*\prime}\right| < \frac{q_1}{2}$$

since, however, up to the maximum of $G^{*\prime}$ there must be $\left|G^{*\prime}\right| < (1 - F^{*\prime})$, there is obtained for $\zeta = \zeta_0$

$$\left|1 - F^{*\prime}\right| > G^{*\prime} > \frac{q_1}{2}$$

that is, a contradiction arises.

(b) If G_L', decreasing monotonically, approaches a positive asymptote $q_1 > 0$, the same is valid for the function $q(G_L)$, explained by the transformation (7.8), for $G_L \to \infty$. Thus $\dot{q}$ is negative and tends toward zero for $G_L \to \infty$. Likewise $(-q^2 + p^2)$ becomes <0 for sufficiently large G_L so that, on the basis of the second differential equation of (7.9), $\ddot{q}$ must be positive for sufficiently large values of G_L; this causes $\ddot{q} \to 0$. Then q also must tend toward zero, since otherwise the last named differential equation would yield a contradiction.

Thus it is proved completely that the pair of solutions F_L', G_L' satisfy the boundary conditions at infinity and is therefore the solution of the boundary-value problem. In particular, it is recognized that $(1 - F_L')$, monotonically decreasing, approaches the asymptote 0 whereas G' at first increases up to a maximum and then, likewise, monotonically decreasing, tends toward zero. Moreover, there results that $G_L \to$ Constant for $\zeta \to \infty$. The latter fact can be demonstrated indirectly, with use of the transformation (7.8) and of the differential equations (7.9) for p and q, as follows:

If it is assumed that $G_L \to \infty$ for $\zeta \to \infty$, there follows from the first of equations (7.9), because $\dot{p} < 0$ and $\dot{q} \to 0$ that, from a certain point onward, necessarily $(q\ddot{p} - 2p) > 0$. In particular, $\ddot{p}$ is then always positive, and as a result $\ddot{p} \to 0$. The inequality $(q\ddot{p} - 2p) > 0$ yields therefore, for sufficiently large G_L

$$2p < q\ddot{p} < q \qquad p^2 < \frac{1}{4} q^2 < \frac{1}{2} q^2$$

Because $p < q$, $\dot{q} > 0$ there is obtained from the second equation of (7.9) the statement that, for sufficiently large G_L, $\ddot{q}$ is always positive; that is, $\ddot{q} \to 0$ must also apply. If the second equation of equations (7.9) is now written in the following form:

$$q^2\left(\ddot{q} - \frac{1}{2}\right) + q(cG_L + \dot{q})\dot{q} - \frac{1}{2} \cdot q^2 + p^2 = 0$$

it is recognized immediately that a contradiction in sign arises for $G_L \to \infty$; thus, only $G_L \to$ Constant can apply.

Simultaneously there is obtained quite generally the statement that the boundary-value problem (7.5), under the additional assumption $(1 - F') > 0$, has for $\zeta \to \infty$ only such monotonic solutions which satisfy the additional boundary condition $G \to$ Constant.[11]

Since in the proof of theorem (6) the continuousness of the function $A^*(B)$ for $B = B^*$ was used, an additional remark is required for the case where $A^*(B)$ is discontinuous.

If $\bar{A}^*$ is defined as the lower limit of all A-values (for fixed B) which are $\geqq A^*$ and for which the pertaining solution F' is of the type indicated in theorem (1), the use of the auxiliary theorem 3, (1) ("continuous dependence of the ordinate value of a turning point on the initial conditions") results in an immediate contradiction of the assumption of a discontinuousness of $A^*(B)$, if $A^* = \bar{A}^*$. If, in contrast, $\bar{A}^* > A^*$, there are obtained, using the auxiliary theorem mentioned, only the following statements:

(a) For $A > \bar{A}^*$, all pertaining solutions F' are of the type described in theorem (1).

(b) For $A^* \leqq A \leqq \bar{A}^*$, all pertaining solutions of A are of the type mentioned in theorem (3) (which follows immediately from the definition of A^* and $\bar{A}^*$).

(c) For $A^* - \epsilon$, for sufficiently small ϵ, all pertaining solutions F' are of the type described in theorem (2).

(d) The $A^*(B)$ values lying to the left and right of the considered point B^* can cluster only against A-values of the interval $A^* \leqq A \leqq \bar{A}^*$.

---

[11] The case still to be considered, for proving this contention to be correct: That $G' < 0$ and, monotonically increasing, tends $\to 0$, leads in the first equation of equations (7.5) immediately to a contradiction.

Assume H_1 (or H_2) to be any limit point (cluster point, situated on the straight line $B = B^*$) for A^*-values which lies to the left (or right) of the straight line $B = B^*$.[12] If the solution G_{H_1}' pertaining to the initial values (H_1, B^*) always remains positive, assume $A_L = H_1$. If, on the other hand, G_{H_1}' assumes negative values and if $H_1 < H_2$ (or $H_1 > H_2$), assume A_L to be the upper (or lower) limit of all pairs of initial values (A, B^*) with A within $H_1 \leqq A \leqq H_2$ (or $H_2 \leqq A \leqq H_1$), whose pertaining solution G' becomes negative.[13] The pair of solutions F_L', G_L' pertaining to (A_L, B^*) is then the solution of the boundary-value problem which can be proved in exactly the same manner as in the case of the continuousness of $A^*(B)$.

(c) On the Numerical Solution of the Boundary-Value Problem

The boundary-value problem (6.5) - to which leads the calculation of "similar" boundary layers on bodies of revolution of a particular shape which rotate about their axis in a fluid otherwise at rest - was solved for the values $m = 0.8$ and $m = 1.5$, or $c = 2.125$ and $c = 1.8333$ (the shape of the pertaining bodies of revolution may be seen in fig. 14) by numerical integration according to the Adams interpolation method, starting from the given boundary values for $\zeta = 0$ and estimated values of $F''(0) = A$ and $G''(0) = B$. The values of A and B were varied until a solution resulted which showed the correct asymptotic behavior for large values of ζ.[14]

In detail, the procedure was as follows: First, the coefficients of the power-series expansion were determined. For the first 10 coefficients the following values resulted:

---

[12] Such a cluster point exists since the A^*-values remain limited when approaching the straight line $B = B^*$. (Otherwise, the auxiliary theorem 3, (1) would yield a contradiction.)

[13] The solution G' pertaining to (H_2, B^*) cannot assume any negative values (this is shown by application of the theorem regarding the continuous dependence of the solutions on the initial conditions to a sequence of A^* values of increasing density at H_2), so that the existence of the upper (or lower) limit mentioned is insured.

[14] For $m = 1$ $(c = 2)$, the boundary-value problem had been solved earlier by Cochran (ref. 7), by combination of a power-series expansion in $\zeta = 0$ and an asymptotic expansion for $\zeta = \infty$.

$a_1 = 1$

$a_2 = A$

$a_3 = 0$

$a_4 = 2B$

$a_5 = (4 - c)AB - 2$

$a_6 = (c - 10)A$

$a_7 = (2c - 20)A^2 + (20 - 14c)B^2$

$a_8 = (60 - 62c + 11c^2)AB^2 + (48c - 80)B + (2c - 20)A^3$

$a_9 = (422c - 39c^2 - 600)AB + 80 - 48c$

$a_{10} = (1000 - 662 \cdot c + 39c^2)A + (1280c - 128c^2 - 1800)A^2B +$

$(600 - 904c + 358c^2)B^3$

$b_1 = 0$

$b_2 = B$

$b_3 = -1$

$b_4 = -2A$

$b_5 = (2 - c)B^2$

$b_6 = (4c - 10)B$

$b_7 = (16c - 40)AB + (10 - 4c)$

$b_8 = (20 - 32c + 11c^2)B^3 + (32c - 60)A^2B + (80 - 32c)A$

$b_9 = (318c - 90c^2 - 300)B^2 + (300 - 128c)A^2$

$b_{10} = (1000 - 960c + 256c^2)B + (1928c - 486c^2 - 1800)AB^2 + (600 - 256c)A^3$

F and G were expressed in the form

$$F = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu!} \zeta^\nu \quad \text{and} \quad G = \sum_{\nu=1}^{\infty} \frac{b_\nu}{\nu!} \zeta^\nu$$

With the aid of the power-series expansion the approximate variation of the solutions for different values of A and B in the interval $0 \leqq \zeta \leqq 1.5$ were then ascertained. Thus a first approximation resulted for those values of A and B whose pertaining solutions F', G' show the correct asymptotic behavior for $\zeta \to \infty$.

A further improvement of the initial values of F" and G" was then obtained by calculation of several solutions according to the Adams interpolation method up to the point $\zeta = 4$. (The initial values required for application of the method were determined with the aid of the power series.) Starting from a first approximation $A_1 = 0.5$, $B_1 = 0.5$, the solutions pertaining to the three pairs of values

$$(A_1;B_1) \qquad (A_1 + 0.1;B_1) \qquad (A_1;B_1 + 0.1)$$

were calculated. On the basis of the qualitative behavior of the calculated solutions, a new pair of values $(A_2;B_2)$ was estimated and the solutions F' and G' pertaining to

$$(A_2;B_2) \qquad (A_2 + 0.02;B_2) \qquad (A_2;B_2 + 0.02)$$

were determined. By continued iteration of this method it was possible to ascertain the initial values of F" and G" within an accuracy of about two units in the third decimal place.

To further improve the accuracy, an extension of the numerical integration at least up to the point $\zeta = 6$ would have become necessary, causing an increase in the number of digits; thus the numerical expenditure would have grown so much that it could no longer have been handled by the use of desk calculators. Therefore, use of the asymptotic expansion of F' and G' was also made for further improvement of the desired solution.

If the differential equations are entered into with the expression

$$F' = pf_1' + p^2f_2' + p^3f_3' + \ldots$$

$$G = g + pg_1 + p^2g_2 + p^3g_3 + \ldots$$

$$G' = pg_1' + p^2g_2' + p^3g_3' + \ldots$$

and the coefficients are compared with respect to powers of the arbitrary parameter p, there are obtained for the functions f_ν, g_ν the following differential equations:

$$f_1''' + kf_1'' = 0$$

$$g_1''' + kg_1'' = 0 \qquad (k = c \cdot g)$$

$$f_\alpha''' + kf_\alpha'' = \sum_{\mu=1}^{\alpha-1} \left\{ -cg_\mu f_{\alpha-\mu}'' + 2q_\mu' f_{\alpha-\mu}' \right\}$$

$$= \sum_{\mu=1}^{\alpha-1} \left\{ -c \frac{b_\mu a_{\alpha-\mu}}{\mu^2 k^2} + 2 \frac{a_\mu b_{\alpha-\mu}}{\mu(\alpha - \mu)k^2} \right\} \cdot e^{-\alpha k\zeta} \qquad (\alpha \geqq 2)$$

$$g_\alpha''' + kg_\alpha'' = \sum_{\mu=1}^{\alpha-1} \left\{ -cg_\mu g_{\alpha-\mu}'' + g_\mu' g_{\alpha-\mu}' - f_\mu' f_{\alpha-\mu}' \right\}$$

$$= \sum_{\mu=1}^{\alpha-1} \left\{ -c \frac{b_\mu b_{\alpha-\mu}}{\mu^2 k^2} + \frac{b_\mu b_{\alpha-\mu}}{\mu(\alpha - \mu)k^2} - \frac{a_\mu a_{\alpha-\mu}}{\mu(\alpha - \mu)k^2} \right\} \cdot e^{-\alpha k\zeta}$$

the solution of which reads:

$$f_\alpha'' = a_\alpha e^{-\alpha k\zeta} \qquad f_\alpha' = -\frac{a_\alpha}{\alpha k} e^{-\alpha k\zeta}$$

$$g_\alpha'' = b_\alpha e^{-\alpha k\zeta} \qquad g_\alpha' = -\frac{b_\alpha}{\alpha k} e^{-\alpha k\zeta} \qquad g_\alpha = \frac{b_\alpha}{\alpha^2 k^2} e^{-\alpha k\zeta}$$

with

$$a_\alpha = \frac{1}{k^3(\alpha - 1)} \sum_{\mu=1}^{\alpha-1} \left\{ \frac{b_\mu a_{\alpha-\mu}}{\mu} \left(\frac{c}{\mu} - \frac{2}{\alpha - \mu} \right) \right\}$$

$$b_\alpha = \frac{1}{k^3(\alpha - 1)} \sum_{\mu=1}^{\alpha-1} \left\{ \frac{b_\mu b_{\alpha-\mu}}{\mu} \left(\frac{c}{\mu} - \frac{1}{\alpha - \mu} \right) + \frac{a_\mu a_{\alpha-\mu}}{\mu(\alpha - \mu)} \right\}$$

The first three coefficients of a_ν, b_ν have the following values:

$$a_1$$

$$a_2 = \frac{1}{k^3}(c - 1)a_1 b_1$$

$$a_3 = \frac{a_1}{8k^6} \left\{ (c - 4)a_1^2 + (5c^2 - 17c + 12)b_1^2 \right\}$$

$$b_1$$

$$b_2 = \frac{1}{k^3} \left\{ a_1^2 + (c - 1)b_1^2 \right\}$$

$$b_3 = \frac{b_1}{8k^6} \left\{ (gc - 12)a_1^2 + (5c^2 - gc + 4)b_1^2 \right\}$$

A crude approximation of the constants ε_1, b_1, k remaining free in the asymptotic expansion, on the basis of the calculations already carried out, showed that at the point $\zeta = 4$ it is sufficient for obtaining an accuracy of three decimals to consider the first term of the asymptotic expansions; that is, F' and G' have there a purely

exponential character within the scope of the desired accuracy. Making use of this fact, the further improvement of the initial values of the second derivatives was then taken up.

The result of the calculations for the cases $m = 0.8$ and $m = 1.5$ is compiled in the following table. For comparison, the values for $m = 1$ also were indicated.[15]

ζ	m = 0.8			m = 1.0			m = 1.5		
	F'	G	G'	F'	G	G'	F'	G	G'
0.0	1.000	0.000	0.000	1.000	0.000	0.000	1.000	0.000	0.000
.1	.938	.002	.046	.939	.002	.046	.939	.002	.047
.2	.877	.009	.083	.878	.009	.084	.879	.009	.085
.3	.818	.019	.112	.819	.019	.114	.820	.019	.115
.4	.761	.031	.135	.762	.031	.136	.764	.032	.138
.5	.706	.046	.152	.708	.046	.154	.709	.046	.156
.6	.654	.061	.164	.656	.062	.166	.658	.063	.169
.7	.605	.078	.172	.607	.079	.174	.609	.080	.177
.8	.558	.096	.176	.561	.097	.179	.563	.098	.182
.9	.514	.113	.178	.517	.115	.181	.520	.116	.184
1.0	.474	.131	.177	.476	.133	.180	.480	.135	.184
1.2	.401	.166	.170	.404	.168	.173	.408	.171	.178
1.4	.338	.199	.159	.341	.202	.162	.345	.206	.167
1.6	.284	.229	.145	.288	.233	.143	.292	.238	.154
1.8	.238	.257	.130	.242	.262	.133	.247	.267	.139
2.0	.199	.281	.115	.203	.286	.118	.208	.294	.124
2.2	.167	.302	.100	.171	.308	.104	.175	.317	.110
2.4	.139	.321	.087	.143	.328	.091	.148	.338	.096
2.6	.116	.338	.075	.120	.345	.078	.124	.355	.083
2.8	.097	.352	.064	.101	.360	.068	.105	.371	.072
3.0	.081	.363	.055	.084	.373	.058	.088	.384	.062
3.2	.068	.374	.047	.071	.384	.050	.074	.396	.053
3.4	.056	.383	.040	.059	.393	.042	.062	.406	.045
3.6	.047	.390	.034	.050	.401	.036	.053	.414	.038
3.8	.039	.396	.028	.042	.408	.030	.044	.421	.032
4.0	.032	.401	.024	.035	.413	.026	.037	.427	.027

[15]The numerical values for $m = 1$ stem from a numerical integration according to the Adams interpolation method, carried out with the initial values $A = -0.616$, $B = 0.510$ indicated in reference 27, p. 75.

The initial values of the second derivatives resulted as follows:

$F''(0) = -0.6186$ $\quad F''(0) = -0.616$ $\quad F''(0) = -0.6125$

$G''(0) = 0.5069$ $\quad G''(0) = 0.510$ $\quad G''(0) = 0.5147$

Consideration of the numerical values yields the result that the influence of the parameter m on the shape of the profiles is extremely small if the deviation from $m = 1$ is not too large.

SUMMARY

The present report deals with the question under what conditions "similar" three-dimensional boundary layers occur, that is, boundary layers for which - if a suitable orthogonal coordinate system is used - both velocity components u and v parallel to the wall in the flow have the property that all u-velocity profiles among them and all v-velocity profiles among them can be made to coincide if u and v as well as the coordinate z perpendicular to the wall are provided with appropriate scale factors which are dependent on the particular wall point. It is then shown that the calculation of such boundary-layer flows may be reduced to the solution of a system of two ordinary nonlinear differential equations of the third order with constant coefficients, under certain boundary conditions. The constancy of the coefficients of the differential equations causes "similar" solutions to occur only when the functions U, V, g_1, g_2 governing the external flow and the metrics of the surface in the flow as well as the scale factor f for the coordinate perpendicular to the wall in the flow satisfy a certain system of partial differential equations. A number of solutions for the latter system of equations are given; however, up till now it has not been possible to prove that no other solutions exist besides those given. Likewise, it is only partially possible to give a simple physical interpretation of the solutions found.

Among the solutions is that boundary-layer flow which appears on bodies of revolution of a particular shape when they rotate about their axis in a fluid otherwise at rest. This case is considered more closely. First, the shape of the bodies of revolution on which "similar" boundary layers occur is ascertained. Then the existence of a solution for the pertaining boundary-value problem is proved, with generalization of the methods developed by R. Iglisch for the two-dimensional case of the differential equation of the wedge flow (application of elementary concepts

of form properties). The numerical solution of the boundary-value problem is carried out for several examples.

Translated by Mary L. Mahler
National Aeronautics and Space Administration

REFERENCES

1. Hayes, W. D.: The Three-Dimensional Boundary Layer. Navord Rep. 1313, Nots 384, May 1951.

2. Howarth, L.: The Boundary Layer in Three-Dimensional Flow. Part I: Derivation of the Equations for Flow Along a General Curved Surface. Phil. Mag., ser. 7, vol. 42, 1951, pp. 239-43.

3. Moore, F. K.: Three-Dimensional Compressible Laminar Boundary-Layer Flow. NACA TN 2279, 1951.

4. Görtler, H.: Zur laminaren Grenzschicht am schiebenden Zylinder. Part I. Archiv der Math., vol. III, 1952.

5. Mangler, W.: Zusammenhang zwischen ebenen und rotationssymmetrischen Grenzschichten. ZAMM, vol. 28, 1948, pp. 97-103.

6. Bödewadt, U. T.: Die Drehströmung über festem Grunde. ZAMM, vol. 20, 1940, pp. 241-248.

7. Cochran, W. G.: The Flow Due to a Rotating Disc. Proc. Cambridge Phil. Soc., vol. XXX, pt. 3, July 1934, pp. 365-375.

8. Hannah, D. M.: Forced Flow Against a Rotating Disc. A.R.C., R. & M., No. 2772, 1952.

9. Howarth, L.: The Boundary Layer in Three-Dimensional Flow. Part II: The Flow Near a Stagnation Point. Phil. Mag., ser. 7, vol. 42, 1951, pp. 1433-40.

10. Howarth, L.: Note on the Boundary Layer on a Rotating Sphere. Phil. Mag., ser. 7, vol. 42, 1951, pp. 1308-15.

11. Moore, F. K.: Laminar Boundary Layer on a Circular Cone in Supersonic Flow at a Small Angle of Attack. NACA TN 2521, 1951.

12. Moore, F. K.: Laminar Boundary Layer on Cone in Supersonic Flow at Large Angle of Attack. NACA Rep. 1132, 1953. (Supersedes NACA TN 2844.)

13. Moore, F. K.: Three-Dimensional Boundary Layer Flow. Jour. Aero. Sci., vol. 20, 1953, pp. 525-534.

14. Moore, F. K.: Displacement Effect of a Three-Dimensional Boundary Layer. NACA TN 2722, 1952.

15. Loos, H. G.: A Simple Laminar Boundary Layer With Secondary Flow. Jour. Aero. Sci., vol. 22, 1955, pp. 35-40.

16. Mager, A.: Three-Dimensional Laminar Boundary Layer With Small Cross-Flow. Jour. Aero. Sci., vol. 21, 1954, pp. 835-845.

17. Mager, A., and Hansen, A.: Laminar Boundary Layer Over Flat Plate in a Flow, Having Circular Streamlines. NACA TN 2568, 1952.

18. Wilkinson, J.: Some Examples of Three-Dimensional Effects in Boundary Layer Flow. The Aeron. Quart., vol. V, 1954, pp. 73-84.

19. Mangler, W.: Die "ähnlichen" Lösungen der Prandtlschen Grenzschichtgleichungen. ZAMM, vol. 23, 1943, p. 243 ff.

20. Prandtl, L.: Über Reibungsschichten bei dreidimensionalen Strömungen. Betz-Festschrift, 1945, pp. 134-141. (Available in English in Reps. and Translations, No. 64, British M.A.P. Völkenrode, May 1, 1946.)

21. Kármán, Theodore Von: Über laminare und turbulente Reibung. ZAMM 1, 1921, p. 233. (Available in English as NACA TM 1092, 1946.)

22. Stuart, J. T.: On the Effects of Uniform Suction on the Steady Flow Due to a Rotating Disc. Quart. Jour. Mech. and Appl. Math., 7, 1954, p. 446.

23. Schuh, H.: Über die "ahnlichen" Lösungen der instationären laminaren Grenzschichtgleichung in inkompressibler Strömung. 50 Jahre Grenzschichtforschung, Verlag Vieweg (Braunschweig), 1955, p. 147.

24. Iglisch, R.: Elementarer Existenzbeweis für die Strömung in der laminaren Grenzschicht zur Potentialströmung $U = u_1 x^m$ mit $m > 0$ bei Absaugen und Ausblasen. ZAMM, vol. 33, 1953, pp. 143-147.

25. Kamke, E.: Differentialgleichungen reeller Funktionen. 2. Auflage, 1950. Akad. Verlagsgesellschaft Leipzig.

26. Kamke, E.: Differentialgleichungen. Lösungsmethoden und Lösungen I, 4. Auflage 1951, Akad. Verlagsges. Leipzig.

27. Schlichting, H.: Grenzschicht-Theorie. Verlag G. Braun (Karlsruhe), 1951.

28. Jahnke, Eugene, and Emde, Fritz: Tafeln höherer Funktionen. Verl. Teubner (Leipzig), 1948.

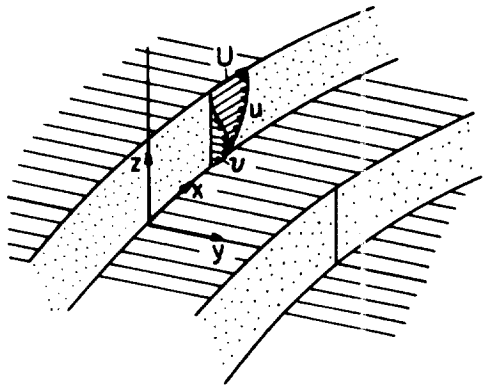


Figure 1.- Appearance of a so-called secondary flow for lateral curvature of the external streamlines.

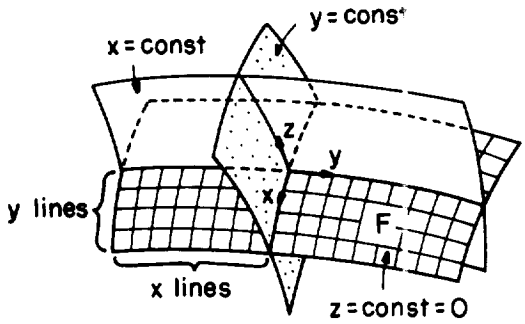


Figure 2.- Regarding the selection of the coordinate system.

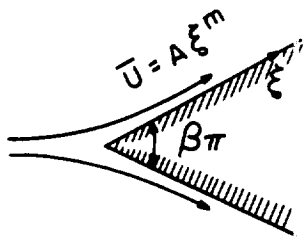


Figure 3.- Flow about a wedge of an opening angle $\beta\pi$.

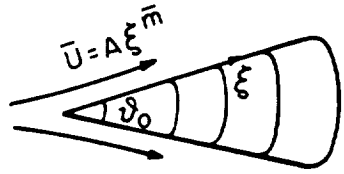


Figure 4.- Axially symmetrical flow about a cone of revolution.

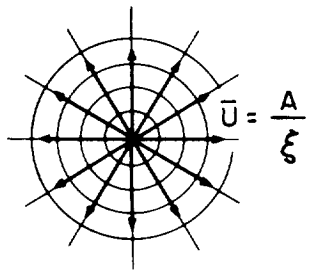


Figure 5.- External flow in the case of a three-dimensional source existing at the zero point.

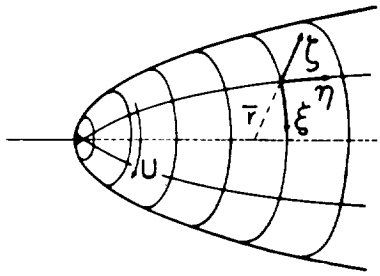


Figure 6.- Rotational flow about a body of revolution.

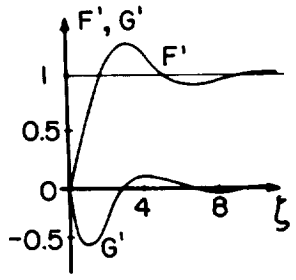


Figure 7.- Velocity profiles for rotational flow over a fixed surface.

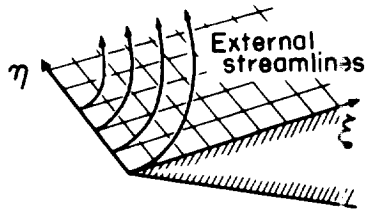


Figure 8.- Flow about a wedge with parabolic external streamlines.

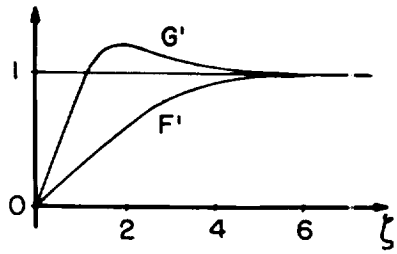


Figure 9.- "Similar" velocity profiles of the components in the direction of the rectangular Cartesian coordinates.

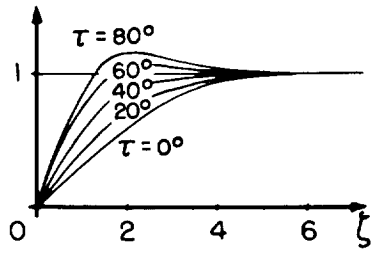


Figure 10.- Velocity profiles of the primary flow component in the direction of the external streamlines.

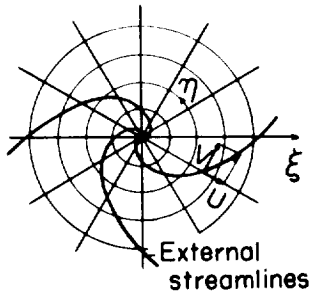


Figure 11.- External streamlines to the velocity distribution (eqs. (5.18)).

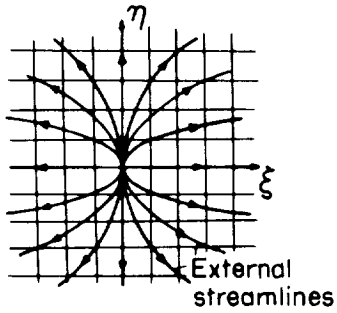


Figure 12.- External flow to the velocity distribution (eqs. (5.4)).

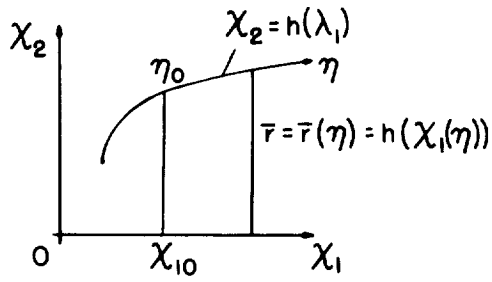


Figure 13.- For determination of the Cartesian equation of the meridian curve.

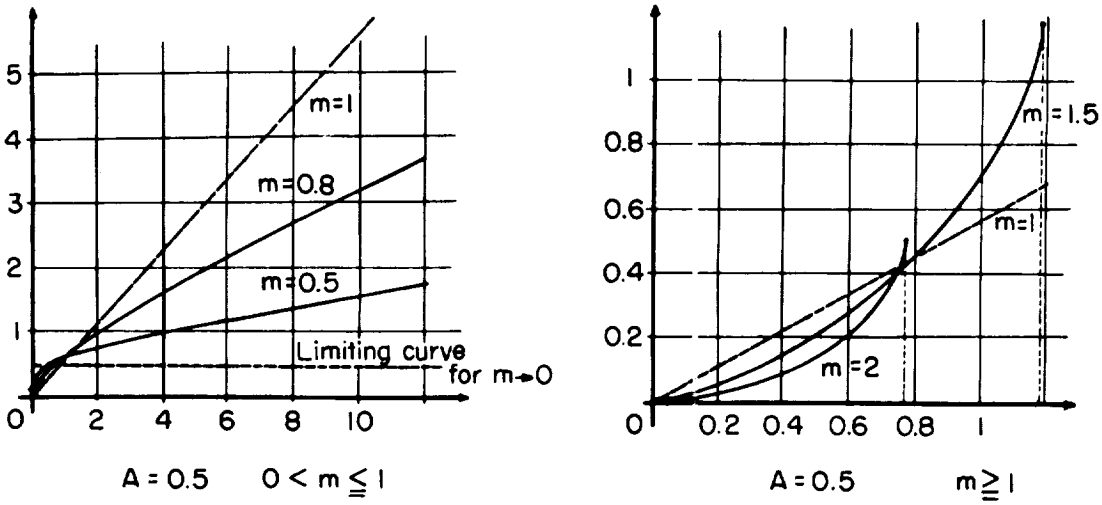


Figure 14.- Meridian curves for various values of m.

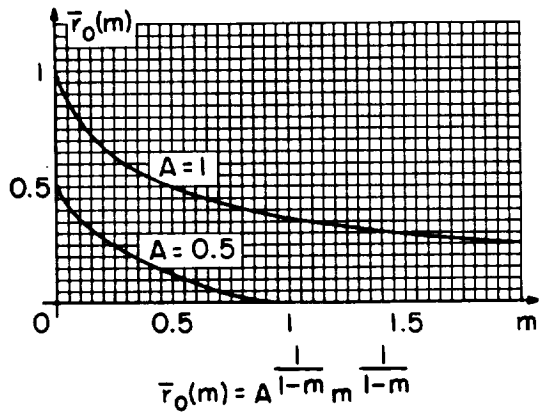


Figure 15.- Radius $\bar{r}_0(m)$ of the opening at the front.

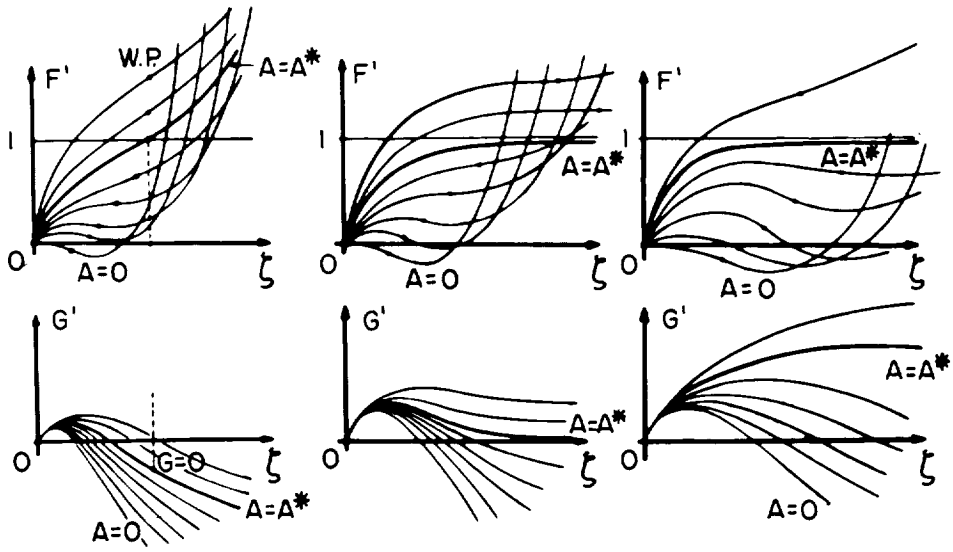


Figure 16.- Qualitative variation of the solution F' and G' for $B < B^*$, $B \approx B^*$, $B > B^*$, and various values of A. (The solution curves given are based partly on numerical calculations, partly on theoretical considerations.)

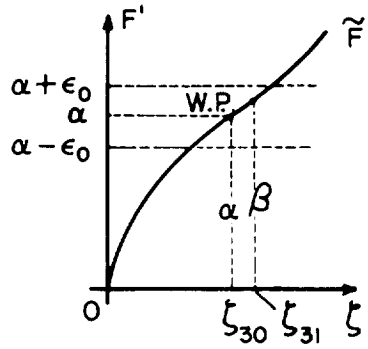


Figure 17.- Explanatory sketch to auxiliary theorem 3,1.

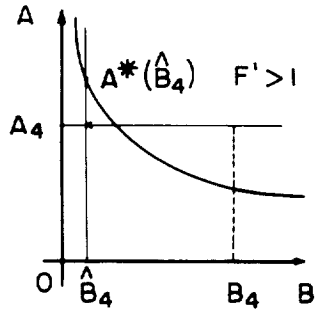


Figure 18.- Explanatory sketch to theorem 4.

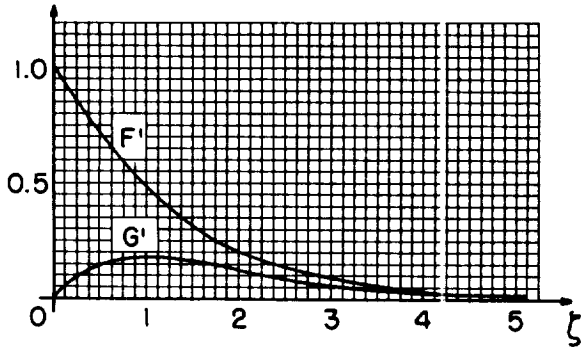


Figure 19.- Velocity profiles for m = 0.8.